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STUDY GUIDE

MATH AI SL

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Mathematics Applications and Interpretations SL Study Guide

Available on www.ib-academy.nl

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Welcome to the IB Academy guide for Mathematics Applications and Interpretations SL.

Our Study Guides are put together by our teachers who worked tirelessly with students and schools. The idea is to compile revision material that would be easy-to-follow for IB students worldwide and for school teachers to utilise them for their classrooms. Our approach is straightforward: by adopting a step-by-step perspective, students can easily absorb dense information in a quick and efficient manner. With this format, students will be able to tackle every question swiftly and without any difficulties.

For this guide, we supplement the new topics with relevant sections from our previous Math Studies, SL and HL study resources, and with insights from our years of experience teaching these courses. We illustrate theoretical concepts by working through IB-style questions and break things down using a step-by-step approach. We also include detailed instructions on how to use the TI-Nspire™ to solve problems; most of this is also quite easily transferable to other GDC models.

The best way to apply what you have learned from the guides is with a study partner. We suggest revising with a friend or with a group in order to immediately test the information you gathered from our guides. This will help you not only process the information, but also help you formulate your answers for the exams. Practice makes better and what better way to do it than with your friends!

In order to maintain our Study Guides and to put forth the best possible material, we are in constant collaboration with students and teachers alike. To help us, we ask that you provide feedback and suggestions so that we can modify the contents to be relevant for IB studies. We appreciate any comments and hope that our Study Guides will help you with your revision or in your lessons. For more information on our material or courses, be sure to check our site at www.ib-academy.nl.

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PRIOR LEARNING

0

0.1 Approximation

0.1.1 Rounding

In math you need to round almost all the time, so its important to know how to do it correctly. To round any number you need to determine the following 2 things:

What does the rounded digit become?

- If the digit is < 5, it stays the same.
- If the digit is ≥ 5, add $+1$ to the digit

e.g. 201.78095
Round to the nearest 10 and 10,000th

Nearest 10
Look at the next digit $\rightarrow 1$
$1 < 5 \Rightarrow 200$

Nearest 10,000th $(= 0.0001)$
Look at the next digit $\rightarrow 5$
$5 \geq 5 \Rightarrow$ add $+1$ to 9 which carries over,
$\Rightarrow 201.7810$.

Which digit is being rounded? (2 possibilities)

- A certain decimal place
  e.g. 201.78095 rounded to:

  $\rightarrow$ 2 decimal places $\Rightarrow 201.78$
  $\rightarrow$ 1 decimal place $\Rightarrow 201.8$

- A certain number of significant figures

  Rule: zeros to the left of the first non-zero digit are not significant

  All other: numbers are significant

  e.g. 0.0023045 rounded to:

  $\rightarrow$ 2 significant figures $\Rightarrow 0.0023$
  $\rightarrow$ 3 significant figures $\Rightarrow 0.00230$
  $\rightarrow$ 4 significant figures $\Rightarrow 0.002305$

The word "estimate" without further detail means "estimate to 1 significant figure". When nothing is specified, always round to 3 significant figures.

0.2 Standard form

Standard form is just a way of rewriting any number, sometimes also referred to as 'scientific notation'. This should be in the form $a \times 10^k$, where a is between 1 and 10, and k is an integer.

10	1×10^1
1000	1×10^3
3280	3.28×10^3
4582000	4.582×10^6

0.3 Sets

0.3.1 Basic set notation

Set any collection of things with a common property (capital letter, curly brackets)

e.g. $A = \{2,4,6,8\}$, $B = \{4, 8\}$, $C = \{\}$, $D = \{1,2,3,4\}$
number of elements in a set: $n(A) = 4$
member of a set: $6 \in A$
empty set: $C = \varnothing$

Subset a set of elements contained in another set $B \subset A$

Intersection elements shared between two or more sets $B \cap D = \{4\}$

Union collection of all elements contained in two or more sets
$B \cup D = \{1,2,3,4,8\}$

0.3.2 Number sets (N, Z, Q and R)

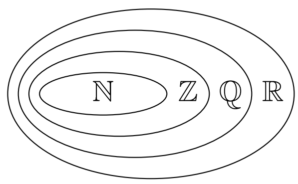
Natural numbers $\mathbb{N} = 0,1,2,3\ldots$

Integers $\mathbb{Z} = \ldots -3,-2,-1,0,1,2,3,\ldots$

Rational numbers $\mathbb{Q}$; all integers and fractions

Real numbers $\mathbb{R}$; all rational and irrational numbers (π, $\sqrt{2}$, etc.)

Sets can be shown in Venn diagrams.



0.4 Algebra and equations

Equation contains an "=" sign.

e.g.
$$\begin{aligned} -2x-3 &= 5 \\ -2x &= 8 \\ x &= -4 \end{aligned}$$

Inequality contains a $>$, $<$, $\geq$ or $\leq$ sign.

e.g. $-2x-3 \geq 5$ ($-2x-3$ is greater than or equal to 5).
Solve like an equation, except if you $\times$ or $\div$ by a negative number, then reverse the inequality!

$$\begin{aligned} -2x-3 &\geq 5 \\ -2x &\geq 8 \\ x &\leq -4 \end{aligned}$$

$0 < a < 1$ means: a is between 0 and 1 (not including 0 and 1)

Absolute value $|x|$ is the positive version of x (distance from 0).

e.g.
$$\begin{aligned} |3| &= 3 \\ |-3| &= 3 \end{aligned}$$

$1 \leq |x| \leq 2$ means: x is between 1 and 2 or between -2 and -1.

0.4.1 Ratios and percentages

Example.

Bryan runs 1.1 km on the first day of his week-long training. Each successive day he runs 20% more than the previous day. How far does Bryan run on the second day?

On the second day he will run 100% (1.1 km) + 20% of 1.1 km, so 120% of 1.1 km.

$$\begin{aligned} \rightarrow 120\% \text{ of } 1.1\,\text{km} &= \frac{120}{100} \times 1.1 \\ &= 1.32\,\text{km} \end{aligned}$$

0.5 Geometry and trigonometry

0.5.1 Lengths, areas and volumes of shapes

These are given in the data booklet

Area of parallelogram	$A = \mathrm{b} \times \mathrm{h}$
Area of a triangle	$A = \frac{1}{2}(\mathrm{b} \times \mathrm{h})$
Area of a trapezium	$A = \frac{1}{2}(\mathrm{a} + \mathrm{b})\mathrm{h}$
Area of a circle	$A = \pi \mathrm{r}^2$
Circumference of a circle	$C = 2\pi \mathrm{r}$
Volume of a pyramid	$V = \frac{1}{3}(\text{area base} \times \text{vertical height})$
Volume of a cuboid (rectangular prism)	$V = \mathrm{l} \times \mathrm{w} \times \mathrm{h}$
Volume of a cylinder	$V = \pi \mathrm{r}^2 \mathrm{h}$
Area of the curved surface of a cylinder	$A = 2\pi \mathrm{rh}$
Volume of a sphere	$V = \frac{4}{3}\pi \mathrm{r}^3$
Volume of a cone	$V = \frac{1}{3}\pi \mathrm{r}^2 \mathrm{h}$

NUMBER AND ALGEBRA

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Table of contents & cheatsheet

Exponents

$$x^1 = x$$
$$x^0 = 1$$
$$x^m \cdot x^n = x^{m+n}$$
$$\frac{x^m}{x^n} = x^{m-n}$$
$$(x^m)^n = x^{m \cdot n}$$
$$(x \cdot y)^n = x^n \cdot y^n$$
$$x^{-1} = \frac{1}{x}$$
$$x^{-n} = \frac{1}{x^n}$$

Logarithms

$$a^x = b \quad \Leftrightarrow \quad x = \log_a b$$

Let $a^x = b$, isolate x from the exponent: $x = \log_a b$

Let $\log_a x = b$, isolate x from the logarithm: $x = a^b$

$$\log_{10} x = \log x$$
$$\log_e x = \ln x$$

Arithmetic: + common difference

$$u_n = n^{\text{th}} \text{ term} = u_1 + (n-1)d$$
$$S_n = \text{sum of } n \text{ terms} = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with $u_1 = 1^{\text{st}}$ term, $d =$ common difference.

Geometric: × common ratio

$$u_n = n^{\text{th}} \text{ term} = u_1 \cdot r^{n-1}$$
$$S_n = \text{sum of } n \text{ terms} = \frac{u_1(1-r^n)}{(1-r)}$$

with $u_1 = 1^{\text{st}}$ term, $r =$ common ratio.

Sigma notation

A shorthand to show the sum of a number of terms in a sequence.

Last value of n

$$\sum_{n=1}^{10} 3n - 1 \longleftarrow \text{Formula}$$

First value of n

e.g.

$$\sum_{n=1}^{10} 3n - 1 = \underbrace{(3 \cdot 1) - 1}_{n=1} + \underbrace{(3 \cdot 2) - 1}_{n=2} + \cdots + \underbrace{(3 \cdot 10) - 1}_{n=10} = 155$$

Simple interest

$I = P \times r \times n$

Where

I	amount of interest
P	principal amount
r	interest rate per annum (as decimal)
n	number of years

Compound interest

$$\text{FV} = \text{PV} \times \left(1 + \frac{r}{100k}\right)^{kn}$$

Where

FV	Future Value
PV	Present Value
r	% interest rate
k	number of compounding periods per year
n	number of years

1.1 Exponents and logarithms

1.1.1 Laws of exponents

Exponents always follow certain rules. If you are multiplying or dividing, use the following rules to determine what happens with the powers.

Example.

$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^m \cdot x^n = x^{m+n}$	$4^5 \cdot 4^6 = 4^{11}$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{3^5}{3^4} = 3^{5-4} = 3^1 = 3$
$(x^m)^n = x^{m \cdot n}$	$\left(10^5\right)^2 = 10^{10}$
$(x \cdot y)^n = x^n \cdot y^n$	$(2 \cdot 4)^3 = 2^3 \cdot 4^3$ and $(3x)^4 = 3^4 x^4$
$x^{-1} = \frac{1}{x}$	$5^{-1} = \frac{1}{5}$ and $\left(\frac{3}{4}\right)^{-1} = \frac{4}{3}$
$x^{-n} = \frac{1}{x^n}$	$3^{-5} = \frac{1}{3^5} = \frac{1}{243}$

1.1.2 Laws of logarithms

Logarithms are the inverse mathematical operation of exponents, like division is the inverse mathematical operation of multiplication.

DB 1.5

$$a^x = b \quad \Leftrightarrow \quad x = \log_a b$$

$\log_a 0 = x$ is always undefined (because $a^x \neq 0$)

This formula shows that the variable x in the power of the exponent becomes the subject of your log equation, while the number a becomes the base of your logarithm.

In your exam you will only work with logarithms with bases of 10 and e. These have special notations in which their base is not explicitly noted.

$$\log_{10} x = \log x$$
$$\log_e x = \ln x$$

You can use logarithms to solve equations with unknowns in the power.

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Solve x in exponents using logarithms

Solve $e^{2x-1} = 250$.

1.	Rearrange into logarithm form.	$2x - 1 = \ln 250$
2.	Make x the subject.	$x = \dfrac{\ln 250 + 1}{2}$
3.	Solve.	$x = 3.26$

1.2 Sequences and series

1.2.1 Arithmetic sequences

Arithmetic sequence the next term is the previous number + the common difference (d).

e.g. $2, 4, 6, 8, 10, \dots \quad d = +2$ and $2, -3, -8, -13, \dots \quad d = -5$

To find the common difference d, subtract two consecutive terms of an arithmetic sequence from the term that follows it, i.e. $d = u_{(n+1)} - u_n$.

Use the following equations to calculate the n^{th} term or the sum of n terms.

$$u_n = u_1 + (n-1)d \qquad S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

with $u_1 = a = 1^{\text{st}}$ term $\qquad d =$ common difference

DB 1.2

Often the IB requires you to first find the 1^{st} term and/or common difference.

Finding the first term u_1 and the common difference d from other terms

In an arithmetic sequence $u_{10} = 37$ and $u_{22} = 1$. Find the common difference and the first term.

1.	Put numbers into n^{th} term formula.	$37 = u_1 + 9d$ $1 = u_1 + 21d$
2.	Equate formulas to find d (using substitution method to solve simultaneous equations).	$21d - 1 = 9d - 37$ $12d = -36$ $d = -3$
3.	Use d to find u_1.	$1 - 21 \cdot (-3) = u_1$ $u_1 = 64$

1.2.2 Geometric sequences

Geometric sequence the next term is the previous number multiplied by the common ratio (r).

$$\text{e.g.} \quad 2, 4, 8, 16, 32, \ldots \quad r = 2 \qquad \text{and} \quad 25, 5, 1, 0.2, \ldots \quad r = \frac{1}{5}$$

To find the common ratio, divide any term of an arithmetic sequence by the term that precedes it, i.e. $r = \dfrac{u_{(n+1)}}{u_n}$

Use the following equations to calculate the n^{th} term or the sum of n terms.

DB 1.3

$$u_n = n^{\text{th}} \text{ term} \qquad S_n = \text{sum of } n \text{ terms}$$

$$= u_1 \cdot r^{n-1} \qquad = \frac{u_1(1-r^n)}{(1-r)}$$

again with

$$u_1 = a = 1^{\text{st}} \text{ term} \qquad r = \text{common ratio}$$

Similar to questions on Arithmetic sequences, you are often required to find the 1$^{\text{st}}$ term and/or common ratio first. Generally the trickiest Geometric sequences questions are those that ask you to find n.

Finding the number of terms n of a geometric sequence

Tracy is training for a long distance swimming competition. She swims 500 m on the day she begins training. Every consecutive day she swims 10% more than the previous day, until the day her training distance first exceeds 3 km. After how many days does Tracy's training first exceed a distance of 3 km?

Questions like this are often asked in relation to finance too, for example when dealing with annual depreciation. You can use geometric sequence formulas to solve them too.

1.	Find r.	An increase of 10% = 110%, so $r = \frac{110}{100} = 1.1$
2.	Put numbers into n^{th} term formula.	$3000 = 500 \times 1.1^{n-1}$
3.	Plot both sides of the equation as separate functions on your GDC and find the x-coordinate of their intersection point.	$x = 19.799$
4.	Use the x-coordinate to find n.	$x = n$ $n = 19.799 \rightarrow 20$ days

1.2.3 Sigma notation

Sigma notation is a way to represent the summation of any sequence — this means that it can be used for both arithmetic or geometric series. The notation shows you the formula that generates terms of a sequence and the upper and lower limits of the terms that you want to add up in this sequence.

Last value of n

$$\sum_{n=1}^{10} 3n-1 \longleftarrow \text{Formula}$$

First value of n

$$\text{e.g. } \sum_{n=1}^{10} 3n-1 = \underbrace{(3\cdot 1)-1}_{n=1} + \underbrace{(3\cdot 2)-1}_{n=2} + \underbrace{(3\cdot 3)-1}_{n=3} + \cdots + \underbrace{(3\cdot 10)-1}_{n=10} = 155$$

1.3 Finance

1.3.1 Simple interest

Simple interest is given by the following formula.

$$I = P \times r \times n$$

where I = amount of interest, P = principal amount, r = interest rate per annum (as decimal), n = no. years

This works like an arithmetic sequence; for each year that interest is counted over a principle sum, a fixed interest rate is charged. This yearly interest is simply a percentage of the principal sum.

Example.

$1500 is invested at 5.25% simple interest per year. How much interest would be earned after 6 years?

$$\begin{aligned} I &= 1500 \times 0.0525 \times 6 \\ &= \$472.50 \end{aligned}$$

1.3.2 Compound interest

Compound interest refers to interest being added to an investment or principle sum every compounding period (instead of e.g. being paid out each time). This means that every time you calculate interest, you do so on a principle sum + previous interest. This works like a geometric sequence.

DB 1.4

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

where

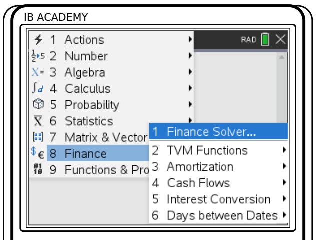
FV	future value
PV	present value
n	no. years
k	no. compounding periods per year
r	% interest rate

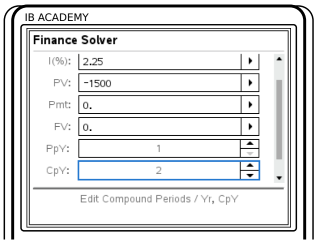
Besides using the equation in your data booklet, you can also use the TVM Solver ("Time Value of Money") on your GDC to solve compound interest questions.

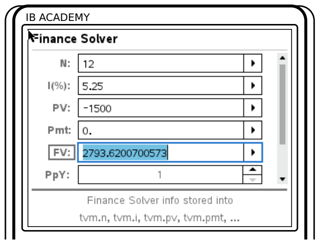
Abbreviation	Stands for
TVM	Time Value of Money
N	Number of payments
I%	percentage Interest rate
PV	Present Value – should be negative
PMT	PayMenT
FV	Future Value
P/Y	Payments per Year
C/Y	Compounding periods per Year

Solving questions about compound interest

$1500 is invested at 5.25% per annum. The interest is compounded twice per year. How much will it be worth after 6 years?







Press (menu), choose
`8: Finance`
`1: Finance Solver`

Enter all known values
For this example:
`N=12` (payments)
`I=5.25` (interest rate)
`PV=-1500` (present value)
negative because investment represents cash outflow;
`PMT=0`
`FV=0` (future value)
`P/Y=1` (payment/yr)
`C/Y=2` (compound/yr)

Highlight cell of asked value, in this case `FV`,
press (enter)

So FV = $2793.62

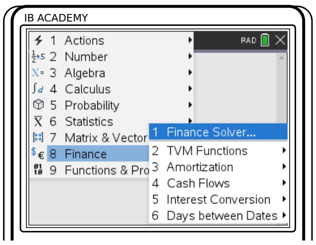
1.3.3 Annuities and amortization

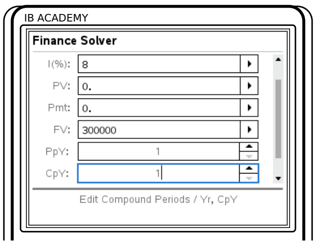
Annuity a series of equal cash flows over equal periods in time

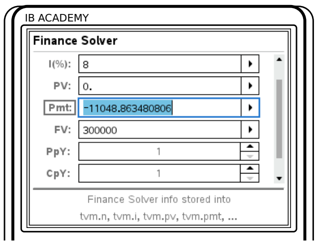
Amortization the process of spreading out a payment into a series of equal instalments over time

Finding the payment amount of an annuity

Jerome would like to save \$300,000 to buy an apartment in 15 years' time. If he can invest at an 8% interest rate per year, how much money would he need to invest at the end of each year to reach his goal?







Press menu, choose
`8: Finance`
`1: Finance Solver`

Enter all known values
For this example:
N=15 (years)
I=8 (interest rate)
PV=0 (present value)
PMT=0
FV=300000 (future value)
P/Y=1 (payment/yr)
C/Y=1 (compound/yr)

Highlight cell of asked value, in this case PMT,
press enter

$\text{PMT} = -\$11{,}048.86$, so he would need to invest this amount each year

1.4 Estimation and error

Error by how much an estimate differs from the actual value

This can be found by calculating the approximate value − exact value

$$V_A - V_E$$

Percentage error $\left|\dfrac{\text{approximate value} - \text{exact value}}{\text{exact value}}\right| \times 100$

$$\left|\frac{V_A - V_E}{V_E}\right| \times 100$$

DB 1.6

Example.

John estimates a 119.423 cm piece of plywood to be 100 cm. What is the error?

$$\begin{aligned}\text{Error} &= V_A - V_E \\ &= 100 - 119.423 \\ &= -19.423 \approx -19.4\end{aligned}$$

What is the percentage error?

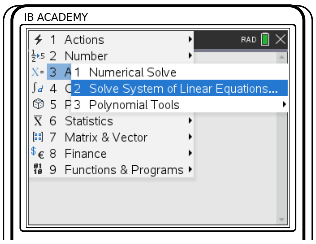
$$\begin{aligned}\text{Percentage error} &= \left|\frac{100 - 119.423}{119.423}\right| \times 100 \\ &= |-0.1626| \times 100 \\ &= 0.1626 \times 100 \approx 16.3\%\end{aligned}$$

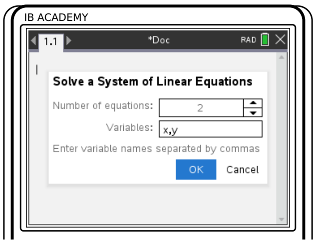
1.5 Simultaneous equations

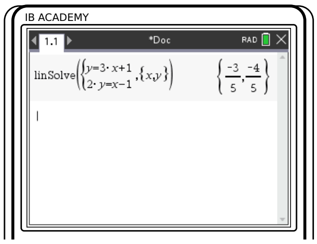
In algebraic problems where you have two unknown variables, for example x and y, and two equations, you can solve for x and y simultaneously. The easiest way is to use the Simultaneous Equation Solver on your GDC.

Solving systems of linear equations

$y = 3x + 1$ (1) and $2y = x - 1$ (2); find the values of x and y







Press menu, choose
`3: Algebra`
`2: Solve System of Linear Equations`

`Number of equations = 2`
Press OK

Enter the two equations
press enter

So $x = -\dfrac{3}{5}$ and $y = -\dfrac{4}{5}$

You will be expected to solve systems of up to 3 linear equations with this method.

In case you prefer to solve a pair of simultaneous equations by hand, there are two methods you can use.

Example.

Elimination

Multiply an equation and then subtract it from the other in order to eliminate one of the unknowns.

$$3 \times (2) \Rightarrow \quad (3) \quad 6y = 3x - 3$$

$$(3) - (1) \Rightarrow \quad 6y - y = 3x - 3x - 3 - 1$$

$$5y = -4$$

$$y = -\frac{4}{5}$$

Put y in (1) or (2) and solve for x

$$-\frac{4}{5} = 3x + 1$$

$$3x = -\frac{9}{5}$$

$$x = -\frac{9}{15} = -\frac{3}{5}$$

Substitution

Rearrange and then substitute one in to another.
Substitute (1) into (2)

$$2(3x + 1) = x - 1$$

$$6x + 2 = x - 1$$

$$5x = -3$$

$$x = -\frac{3}{5}$$

Put x in (1) or (2) and solve for x

$$y = 3\left(-\frac{3}{5}\right) + 1$$

$$y = -\frac{4}{5}$$

FUNCTIONS

2

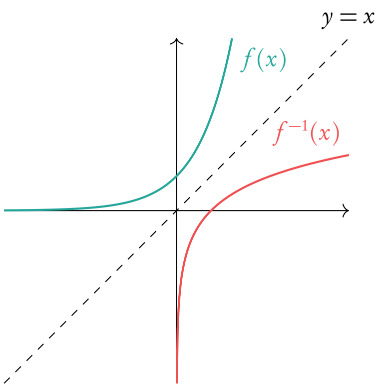
Table of contents & cheatsheet

Definitions

Function a mathematical relationship where each input has a single output. It is often written as $f(x)$ where x is the input
Domain all possible x values, the input. (the domain of investigation)
Range possible y values, the output. (the range of outcomes)
Coordinates uniquely determines the position of a point, given by (x, y)

2.1. Basic concepts 20

Inverse function, $f^{-1}(x)$ reflection of $f(x)$ in $y = x$.



Transforming functions

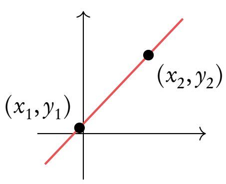
Change to $f(x)$	Effect
$f(x)+a$	Move graph a units upwards
$f(x+a)$	Move graph a units to the left
$a \cdot f(x)$	Vertical stretch by factor a
$f(a \cdot x)$	Horizontal stretch by factor $\frac{1}{a}$
$-f(x)$	Reflection in x-axis
$f(-x)$	Reflection in y-axis

2.2. Linear functions 22

$y = mx + c$

y-intercept: c

Gradient: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$



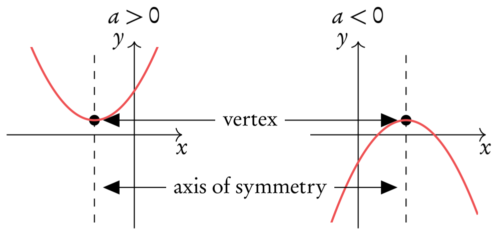
Parallel lines: $m_1 = m_2$ (same gradients)
Perpendicular lines: $m_1 m_2 = -1$

2.3. Quadratic functions 25

$y = ax^2 + bx + c = 0$

Axis of symmetry: x-coordinate of the vertex: $x = \dfrac{-b}{2a}$

Factorized form: $y = (x+p)(x+q)$
Vertex form: $y = a(x-h)^2 + k$
Vertex: (h, k)



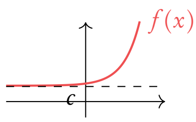
2.4. Polynomials 26

$y = ax^n$

Polynomial: a function that contains one or more terms raised to different powers

2.5. Exponential functions 28

$y = a^x + c$



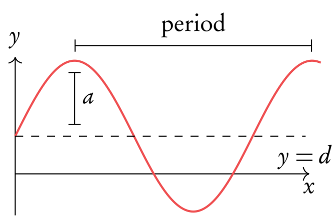
Horizontal asymptote: $y = c$
y-intercept: $k + c$

Asymptote: a line that a graph approaches but never quite touches.

2.6. Sinusoidal functions 29

$y = a\sin(bx) + d = 0$

Amplitude: a
Period: $\dfrac{360°}{b}$
Principle axis: $y = d$



2.1 Basic concepts

2.1.1 Domain and range

A mathematical model transforms an input value into an output value. To describe a mathematical model (or function) you therefore need to know the possible x and y-values that it can have; these are called the domain and the range respectively.

Function a mathematical relationship where each input has a single output. It is often written as $y = f(x)$ where x is the input.

Domain all possible x-values that a function can have. You can also think of this as the 'input' into a mathematical model.

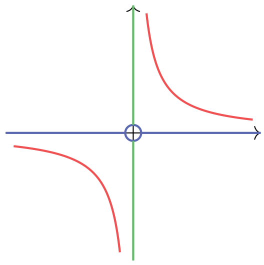
Range all possible y-values that a function can give you. You can also think of this as the 'output' of a mathematical model.

Coordinates uniquely determine the position of a point, given by (x, y).

Note that some questions will specify the domain (often even though the function as such could theoretically have many other x-value inputs). Make sure that your answers are within any given domain; for example, only sketch the graph for the x-values included in the domain if you are asked to draw it.

Example.

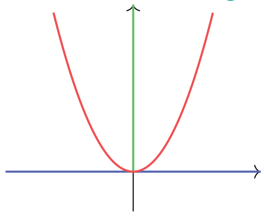
Find the domain and range for the function $y = \dfrac{1}{x}$



Domain: $x \neq 0$
(all real numbers except 0)

Range: $y \neq 0$
(all real numbers except 0)

Find the domain and range for the function $y = x^2$

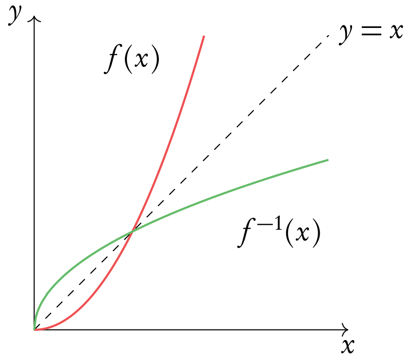


Domain: $x \in \mathbb{R}$
(all real numbers)

Range: $y \in \mathbb{R}^+$
(all positive real numbers)

2.1.2 Inverse functions

Inverse functions are the reverse of a function. Finding the input x for the output y. This is the same as reflecting a graph in the $y = x$ axis.



2.1.3 Sketching graphs

When sketching graphs, make sure your drawing is to scale and within the required domain and range.

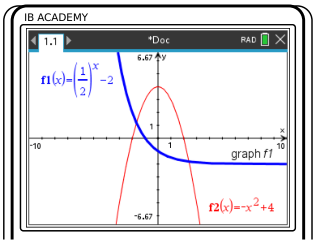
Even though the IB does not not expect your sketches to be completely precise, it is important that key features are in the right place. These include:

- x- and y-intercepts
- intersection points
- turning points
- axes of symmetry
- horizontal and vertical asymptotes

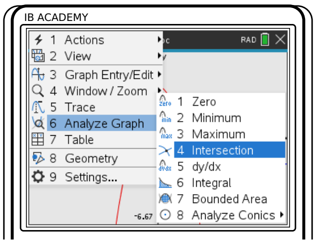
As well as sketching these features, you will need to know how to identify them on given graphs and on your GDC.

Finding a point of intersection

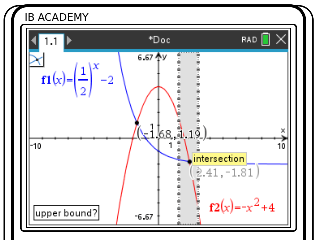
$f(x) = \left(\frac{1}{2}\right)^x - 2$ and $g(x) = -x^2 + 4$. Find the coordinates where $f(x) = g(x)$.



Plot both functions



Press menu
`6: Analyse Graph`
`4: Intersection`



Approach the intersection you are trying to find with the cursor and click once you are near it. Repeat for any other intersections.

The intersection points are $(-1.68, 1.19)$ and $(2.41, -1.81)$. If you are sketching functions $f(x)$ and $g(x)$, be sure to mark and label these points on your sketch.

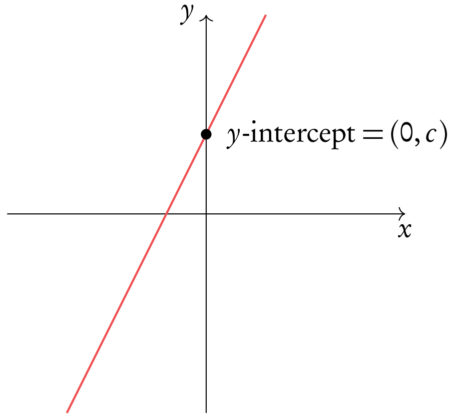
2.2 Linear models

Linear functions make straight line graphs. Two elements you need to know to describe a linear function are its slope/gradient (how steeply it is rising or decreasing) and its y-intercept (the y-value when the function crosses the y-axis, so when $x = 0$).

Straight line equation is usually written in the following form:

$$y = mx + c$$

with
$m =$ gradient (slope)
$c = y$-intercept



This is useful, because this way you can read the gradient (m) and y-intercept (c) directly from the equation (or formulate a straight line equation yourself, if you know the value of the gradient and y-intercept.)

You may also see a straight line equation written in two other forms:

$ax + by + d = 0$	general form
$y - y_1 = m(x - x_1)$	point-slope form

In these cases, it is best to rearrange the equation into the $y = mx + c$ form discussed above. You can do this by using the rules of algebra to make y the subject of the equation.

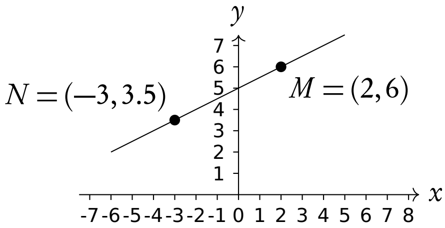
Make sure you substitute the y and x-coordinates in the correct order!

DB 2.1

When you are not given the value of the gradient in a question, you can find it if you know two points that should lie on your straight line. The gradient (m) can be calculated by substituting your two known coordinates (x_1, y_1) and (x_2, y_2) into the following equation:

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Finding the equation of a line and its x-intercept

Find the equation of the straight line that passes through points $M(2,6)$ and $N(-3,3.5)$. Then find the x-intercept.	
1. Take two points on the graph and substitute the values into the formula.	$m = \frac{y_2-y_1}{x_2-x_1} = \frac{6-3.5}{2-(-3)} = 0.5$
2. Fill in one point to find c.	$6 = 0.5(2)+c$ $6 = 1+c$ $c = 5$
3. Write down the equation $y = mx+c$ replacing m and c.	$y = 0.5x+5$
4. To find the x-intercept, solve by isolating x.	$0 = 0.5x+5$ $-5 = 0.5x$ $x = -10$ x-intercept: (-10,0)

2.2.1 Intersection of lines

Finding a linear equation with a given gradient and intersection point

Line L_1 has a gradient of 5 and intersects line L_2 at point $A(1,0)$. Find the equation of L_1

1. Find slope.	Slope given, $m = 5$
2. Fill in one point to find c.	L_1 passes through $(1,0)$ $\Rightarrow 0 = 5(1)+c$ $c = -5$ $\Rightarrow y = 5x-5$

2.2.2 Parallel and perpendicular lines

When you know the equation of one straight line, you can use the value of its gradient, m, to find equations of other straight lines that are parallel or perpendicular to it.

Parallel lines	have the same slope	$m_1 = m_2$
Perpendicular lines	meet at a 90° angle	$m_2 = \frac{-1}{m_1}$

Finding a linear equation of a perpendicular line

Line L_1 has a gradient of 5 and intersects line L_2 at point $A(1,0)$. Line L_2 is perpendicular to L_1. Find the equation of L_2

1. Find slope.

L_2 is perpendicular to L_1 so

$$m = -\frac{1}{\text{gradient}}$$

$$\Rightarrow m = -\frac{1}{5}$$

2. Fill in one point to find c.

$$0 = -\frac{1}{5}(1) + c$$

$$c = \frac{1}{5}$$

$$\Rightarrow y = -\frac{1}{5}x + \frac{1}{5}$$

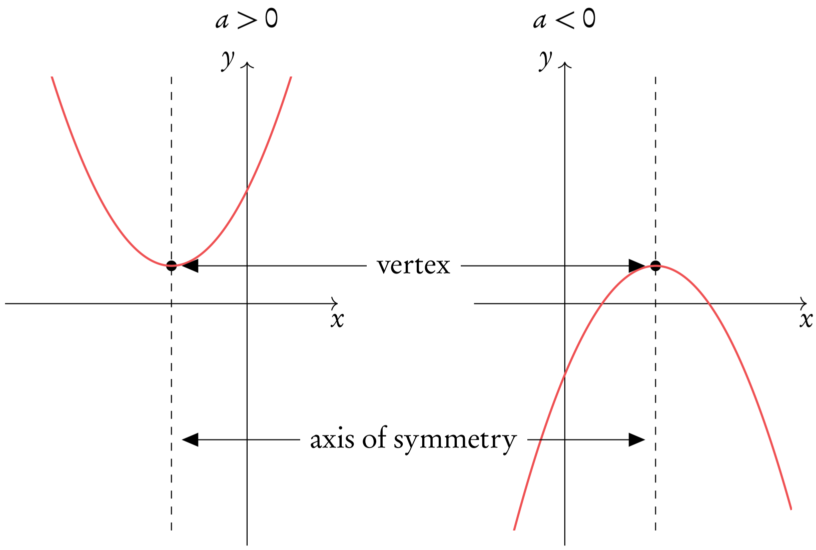
2.3 Quadratic models

A quadratic function consist of an equation with a term raised to the power of 2.

$$y = ax^2 + bx + c = 0$$

When plotted on a graph, a quadratic function makes an upward or downward facing parabolic shape. A parabola always has a vertex (the maximum or minimum point) and an axis of symmetry.

If you know the x and y coordinate of the vertex, the equation for the axis of symmetry will always be $x =$ [the x-coordinate of the vertex]. This also works the other way around; the equation of the axis of symmetry gives you the x-coordinate of the vertex.



The equation for the axis of symmetry can be found using the equation below where a, b and c are the corresponding numbers from your quadratic equation written in the form $y = ax^2 + bx + c$.

Axis of symmetry $\quad x = \dfrac{-b}{2a} = x\text{-coordinate of vertex}$

DB 2.5

Finding the vertex of a quadratic function

Given that $f(x) = x^2 - 2x - 15$, find the coordinates of the vertex of $f(x)$.

1.	Use axis of symmetry formula to find x-coordinate of the vertex.	$x = \dfrac{-b}{2a} = \dfrac{-(-2)}{2 \cdot 1}$ $\Rightarrow x = 1$	
2.	Use $f(x)$ to find y-coordinate of the vertex.	$y = 1^2 - 2(1) - 15$ $\Rightarrow y = -16$	vertex: $(1, -16)$

2.3.1 Solving quadratic functions

To 'solve' a quadratic function, you need to find its x-intercepts. You find these by setting your quadratic equation equal to 0. When $ax^2+bx+c=0$ you can solve for x to find the x-intercepts (or 'roots', or 'solutions' as they are also called interchangeably). Given that quadratic equations have the shape of a parabola, they can have up to two x-intercepts; as you can see when a quadratic equation is plotted, it often crosses the x-axis twice.

There are several methods to find the x-intercepts. In your exam you will primarily use your GDC. Here we work through an example of factorisation.

Factorisation

Using factorisation to find the roots of a quadratic function

Factorise $f(x)=x^2-2x-15$. Hence find the roots of $f(x)=0$.

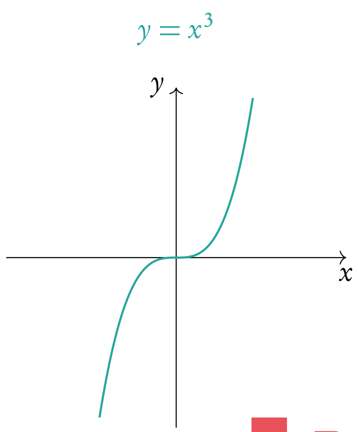
1.	Rearrange $f(x)=0$ into the form $f(x)=(x+p)(x+q)$.	Look for the pair of numbers that satisfy: $\begin{aligned} p+q&=-2 \\ pq&=15 \end{aligned} \Rightarrow \begin{aligned} p&=-5 \\ q&=3 \end{aligned}$ so $f(x)=(x-5)(x+3)$
2.	Solve $x+p=0$ and $x+q=0$ to find the roots (i.e. change the signs on p and q).	$\begin{aligned} x-5&=0 \\ x+3&=0 \end{aligned} \Rightarrow$ roots: $\begin{aligned} x&=5 \\ x&=-3 \end{aligned}$

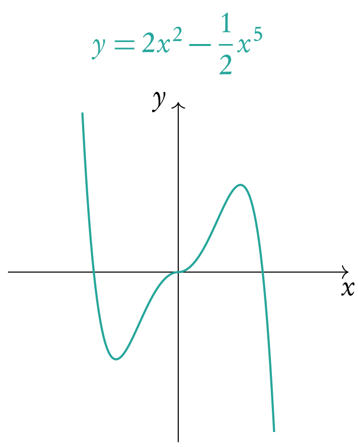
2.4 Polynomials

A polynomial function is made up of one or more summed terms, each of which is generally a variable (e.g. x) raised to a power and multiplied by a coefficient.

$$f(x)=ax^n$$

Example.



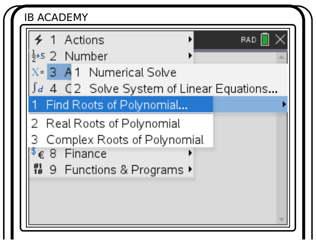


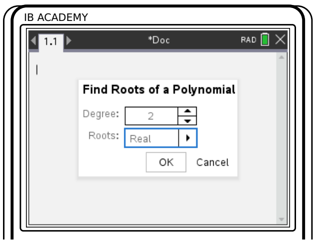
Use GDC to see what a given polynomial looks like.

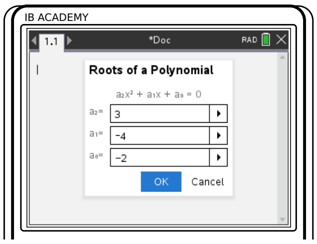
You can use the `Polyrootfinder` on your GDC to find roots of any polynomials.

Solving polynomial equations on GDC

Solve $3x^2 - 4x - 2 = 0$



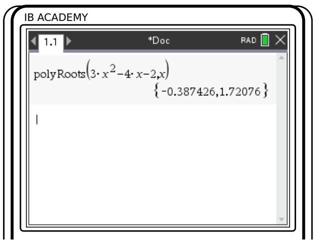




Press menu, choose
`3: Algebra`
`3: Polynomial Tools`
`1: Find Roots of Polynomial`

`Degree = 2`, `Roots = Real`,
press OK

Enter values a2, a1 and a0.
Press OK



so $x = 1.72$ or $x = -0.387$

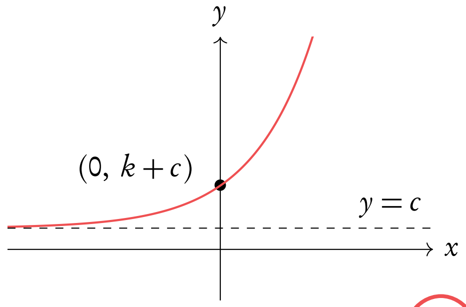
Note that quadratic functions are a special type of polynomial.

2.5 Exponential models

Further, you need to be familiar with exponential functions. An exponential function is one where the variable (e.g. x) is in the power itself.
In questions dealing with exponential functions, you will need to know how to describe their asymptotes and y-intercepts. You can use the components of a function's equation to find these.

$$y = ka^{rx} + c$$

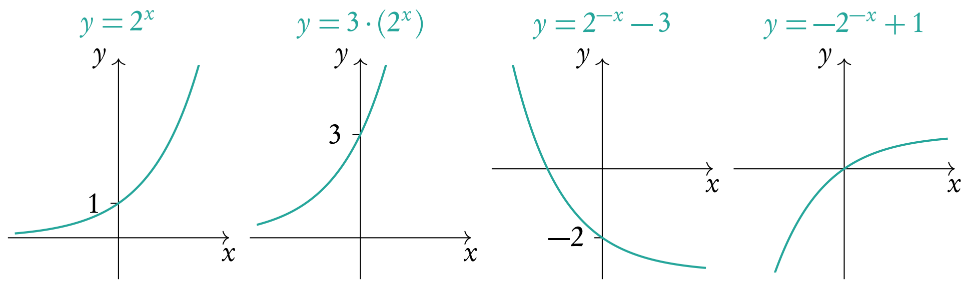
with
horizontal asymptote at $y = c$
y-intercept at $(0, k + c)$



Asymptote a line that a graph approaches but never quite touches.

Use GDC to sketch more complicated functions.

Example.



Solving exponential functions

Remember that e is just a number (2.71828...), so treat it like any other number.

You get points on your IB exam for writing out an equation like this!

A population of rabbits is modelled by the function $P(t) = 24e^{0.19t}$, $t \geqslant 0$, where P is the population of rabbits and t is the time in months. After how many months does the population reach 1 million?

1.	Set up an equation you are looking to solve.	$1000000 = 24e^{0.19t}$
2.	Plot both sides of the equation as separate functions on your GDC.	$y_1 = 1000000$ $y_2 = 24e^{0.19t}$
3.	Find the x-coordinate of the intersection point.	$x = 55.987$ $\Rightarrow$ 56 months

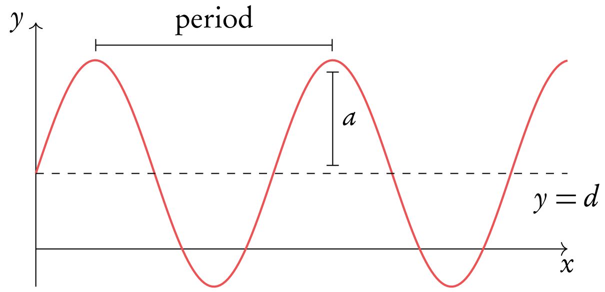
2.6 Sinusoidal models

Trigonometric functions, *sine* and *cosine*, make sinusoidal shapes when graphed.

As with other functions, the height, width and position on the axes of a trigonometric function is determined by its parameters. For sinusoidal models we describe these parameters with special names; the vertical stretch is determined by the amplitude, the horizontal stretch by the period and an upward/downward shift by the position of the principle axis.

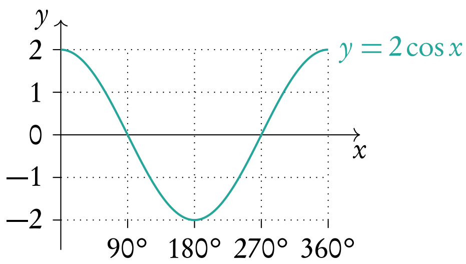
$y = a\sin(bx) + d$

with
amplitude of a
principle axis at $y = d$
period of $\dfrac{360°}{b}$

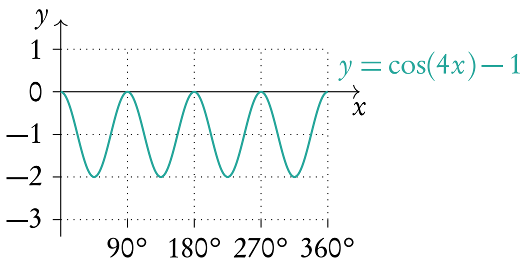


Example.

Transformations of $y = \cos x$.



amplitude $= 2$
principle axis at $y = 0$
period $= \dfrac{360°}{1} = 360°$



amplitude $= 1$
principle axis at $y = -1$
period $= \dfrac{360°}{4} = 90°$

GEOMETRY AND TRIGONOMETRY

3

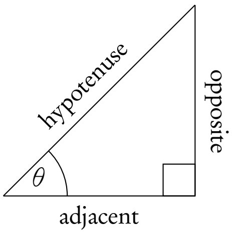
Table of contents & cheatsheet

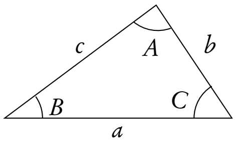
Surface area the sum of the areas of all faces; unit^2

Volume amount of space it occupies; unit^3

$V =$ area of cross-section $\times$ height

$a^2 = b^2 + c^2$	Pythagoras
$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$	SOH
$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	CAH
$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$	TOA





Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

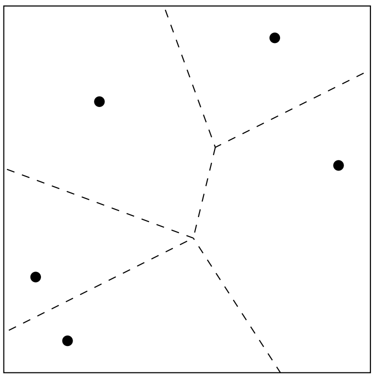
Use this rule when you know: 2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides).

Cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.

Area of a triangle: Area $= \dfrac{1}{2}ab\sin C$

Use this rule when you know: 3 sides or 2 sides and the angle between them.



Site a point on a plane.

Cell a region containing all the points for which the enclosed site is the closest one. Each cell encloses a single site.

Edge a boundary between two cells.

Vertex a meeting point of two or more edges.

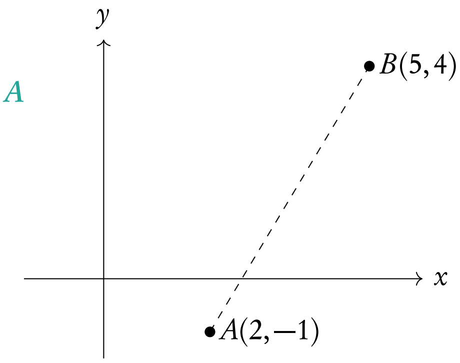
3.1 Lengths, areas and volumes

3.1.1 Distance between points

Example.

Find the distance between points A and B

$$\begin{aligned} d &= \sqrt{(5-2)^2 + (4-(-1))^2} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$



3.1.2 Surface area and volume

Formulas for finding surface areas and volumes are in the data booklet.

DB 3.1

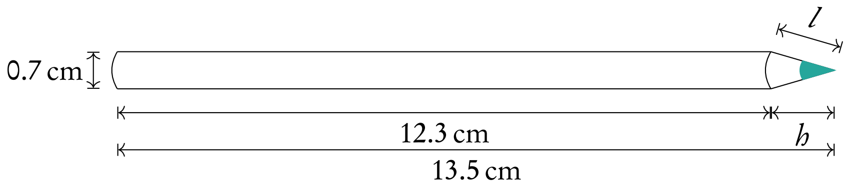
Surface area the sum of the areas of all faces (units^2)

Volume amount of space it occupies (units^3)

$$V = \text{area of cross-section} \times \text{height}$$

Example.

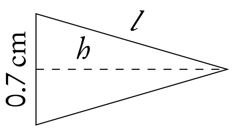
The length of the cylindrical part of a pencil is 12.3 cm



Write down the value of h.

$$h = 13.5 - 12.3$$
$$= 1.2\,\text{cm}$$

Find the value of l.



$$\Rightarrow l^2 = \left(\frac{1}{2} \times 0.7\right)^2 + 1.2^2$$
$$= 1.5625$$
$$\Rightarrow l = 1.25\,\text{cm}$$

Find the total surface area of the pencil.

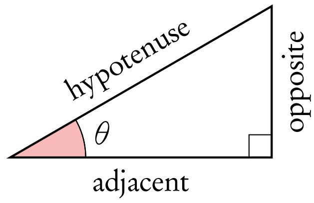
$$SA_{\text{pencil}} = SA_{\text{cylinder}} + SA_{\text{cone}} + SA_{\text{circle}}$$
$$= 2\pi(0.35) \cdot 12.3 + \pi(0.35)(1.25) + \pi(0.35)^2$$
$$= 28.8\,\text{cm}^2$$

Find the volume of the pencil.

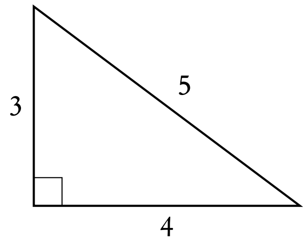
$$V_{\text{pencil}} = V_{\text{cylinder}} + V_{\text{cone}}$$
$$= \pi(0.35^2) \times 12.3 + \frac{1}{3}\pi(0.35)^2 \times 1.2$$
$$= 4.89\,\text{cm}^3$$

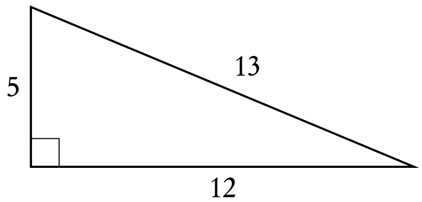
3.2 Right-angled triangles

$a^2 = b^2 + c^2$	Pythagoras' theorem
$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$	SOH
$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	CAH
$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$	TOA



The IB particularly likes asking questions about special right-angled triangles which have whole numbers for all the sides of the right triangles, so it is good to be familiar with the following two triangles.



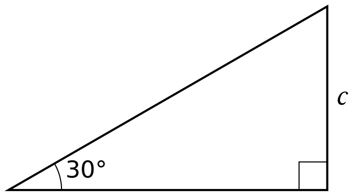


Make sure you read exam questions carefully to determine whether you are looking for an acute (less than 90°) or obtuse (more than 90°) angle. Sometimes they specify this.

To solve problems using Pythagoras, SOH, CAH or TOA identify what information is given and asked. Then determine which of the equations contains all three elements and solve for the unknown.

Finding an angle or the length of a side of a right angle triangle

Find c in the following triangle:



1. Identify:

- info given
- what you need to find

- angle and adjacent given
- need to find opposite

2. Use SOH, CAH, TOA or Pythagoras to find the missing value.

Pythagoras:	3 × length
SOH:	Θ, opp & hyp
CAH:	Θ, adj & hyp
TOA:	Θ, adj & opp

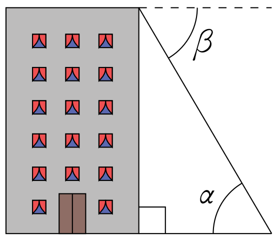
TOA: Θ, adj & opp

$$\tan 30^\circ = \frac{c}{12}$$
$$\Rightarrow c = 12 \times \tan 30^\circ$$
$$= 6.92$$

Some wordy questions will give you information about the sizes of angles in relation to one another. The IB expects you to know what an angle of elevation or depression is.

$\alpha =$ angle of elevation

$\beta =$ angle of depression

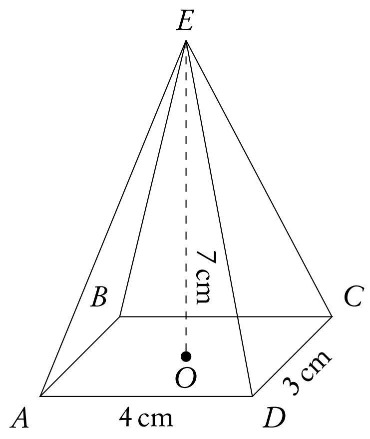


3.2.1 3D applications

To find angles and the length of lines, use *SOH*, *CAH*, *TOA* and *Pythagoras*.

Example.

Rectangular-based pyramid ABCDE with $AD = 4\,\text{cm}$, $CD = 3\,\text{cm}$, $EO = 7\,\text{cm}$.



Find the length of AC.

$$\begin{aligned} AC^2 &= AD^2 + DC^2 \\ &= 4^2 + 3^2 \\ &= 25 \\ \Rightarrow AC &= \sqrt{25} \\ &= 5\,\text{cm} \end{aligned}$$

Find the length of AE.

$$AE^2 = AD^2 + EO^2$$
$$(AO = \frac{1}{2}AC = 2.5)$$
$$\begin{aligned} AE^2 &= 2.5^2 + 7^2 \\ &= 55.25 \\ \Rightarrow AE &= \sqrt{55.25} \\ &= 7.43\,\text{cm} \end{aligned}$$

Find the angle $A\hat{E}C$.

$$A\hat{E}C = 2A\hat{E}O$$
$$\tan A\hat{E}O = \frac{2.5}{7}$$
$$\begin{aligned} \Rightarrow A\hat{E}O &= \tan^{-1}\left(\frac{2.5}{7}\right) \\ &= 19.65^\circ \\ \Rightarrow A\hat{E}C &= 2 \times 19.65 \\ &= 39.3^\circ \end{aligned}$$

Find the angle that AE makes with the base of the pyramid.

Looking for angle $E\hat{A}O$:

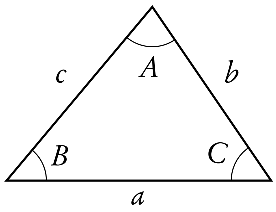
$$\tan E\hat{A}O = \frac{7}{2.5}$$
$$\begin{aligned} \Rightarrow E\hat{A}O &= \tan^{-1}\left(\frac{7}{2.5}\right) \\ &= 70.3^\circ \end{aligned}$$

Find the angle the base makes with EM, where M is the midpoint of CD.

Looking for angle $E\hat{M}O$:

$$\tan E\hat{M}O = \frac{7}{OM}$$
$$(OM = \frac{1}{2}AD = 2\,\text{cm})$$
$$\tan E\hat{M}O = \frac{7}{2}$$
$$\begin{aligned} \Rightarrow E\hat{M}O &= \tan^{-1}\left(\frac{7}{2}\right) \\ &= 74.1^\circ \end{aligned}$$

3.3 Non-right-angled triangles



To find any missing angles or side lengths in non-right angle triangles, use the *sine* and *cosine* rules. Remember that the angles in the triangle add up to 180°.

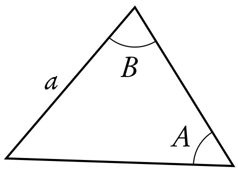
Sine rule

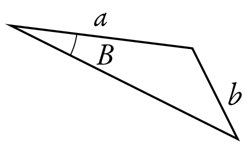
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$ also rearranged as $$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

DB 3.2

Use this rule when you know:

2 angles and a side (not between the angles) or 2 sides and an angle (not between the sides)





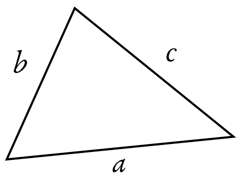
Cosine rule

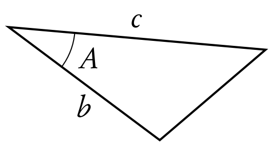
$$a^2=b^2+c^2-2\,bc\cos A$$ also rearranged as $$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

Pick the version of the sine or cosine rule formula depending on whether you are looking for an angle or a side. This will make the algebra easier for you.

Use this rule when you know:

3 sides or 2 sides and the angle between them



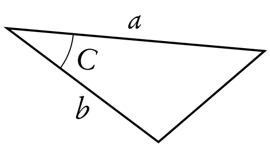


Area of a triangle

$$\text{Area}=\frac{1}{2}ab\sin C$$

Use this rule when you know:

2 sides and the angle between them



Exam tip: make sketches when working with worded questions. This will make it easier for you to pick the right rule.

Example.

$\triangle ABC : A = 40°, B = 73°$, $a = 27$ cm.
Find $\angle C$.

$\angle C = 180° - 40° - 73° = 67°$

Find b.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{27}{\sin 40°} = \frac{b}{\sin 73°}$$
$$b = \frac{27}{\sin 40°} \cdot \sin 73° = 40.169 \approx 40.2\,\text{cm}$$

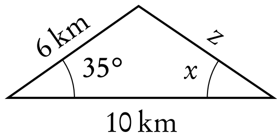
Find c.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$c = \frac{27}{\sin 40°} \times \sin 67° = 38.7\,\text{cm}$$

Find the area.

$$\text{Area} = \frac{1}{2} \cdot 27 \cdot 40 \cdot 2 \cdot \sin 67°$$
$$= 499.59 \approx 500\,\text{cm}^2$$

Example.



Find z.

$$z^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos 35°$$
$$z^2 = 37.70$$
$$z = 6.14\,\text{km}$$

Find $\angle x$.

$$\frac{6}{\sin x} = \frac{6.14}{\sin 35°}$$
$$\sin x = 0.56$$
$$x = \sin^{-1}(0.56) = 55.91°$$

3.4 Circles

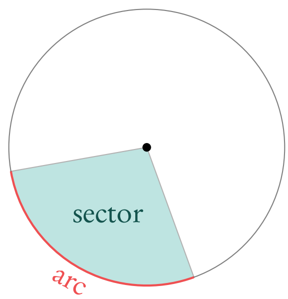
3.4.1 Circle formulas

DB 3.4

$$\textbf{Arc length} = \frac{\vartheta}{360} \times 2\pi r$$

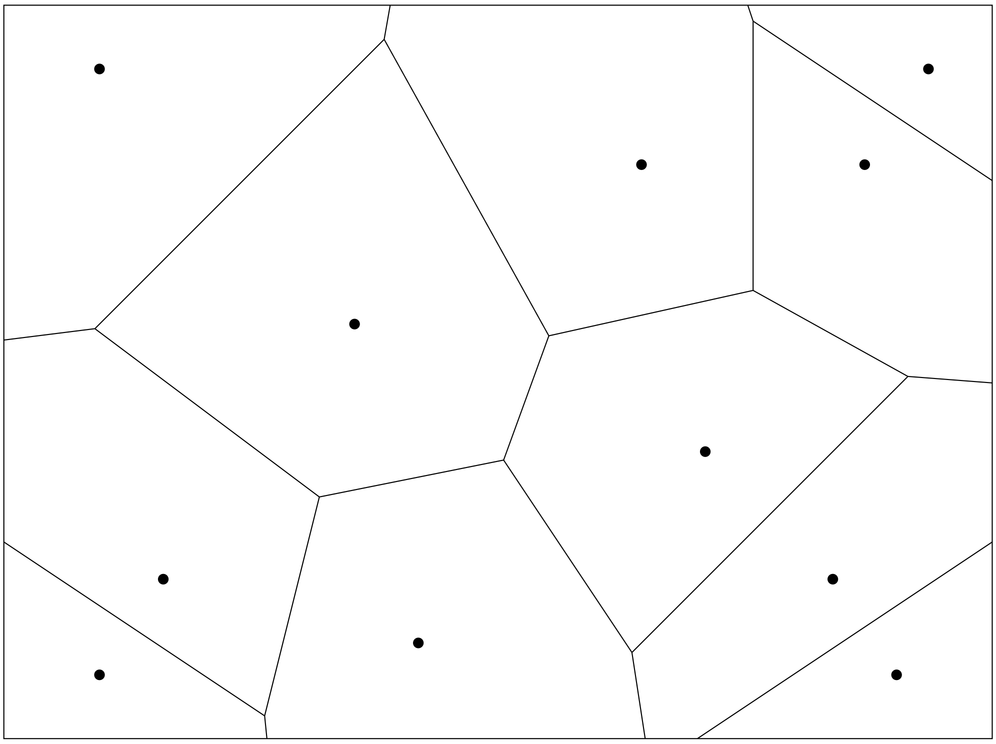
$$\textbf{Area of a sector} = \frac{\vartheta}{360} \times \pi r^2$$

ϑ = angle in degrees, r = radius



3.5 Voronoi diagrams

With a Voronoi diagram you can divide a plane into regions based on a set of sites in it. The partitioning is based on the minimal distance to sites.



Site a point on a plane

Cell a region containing all the points for which the enclosed site is the closest one. Each cell encloses a single site

Edge a boundary between two cells

Vertex a meeting point of two or more edges

3.5.1 Nearest neighbour interpolation

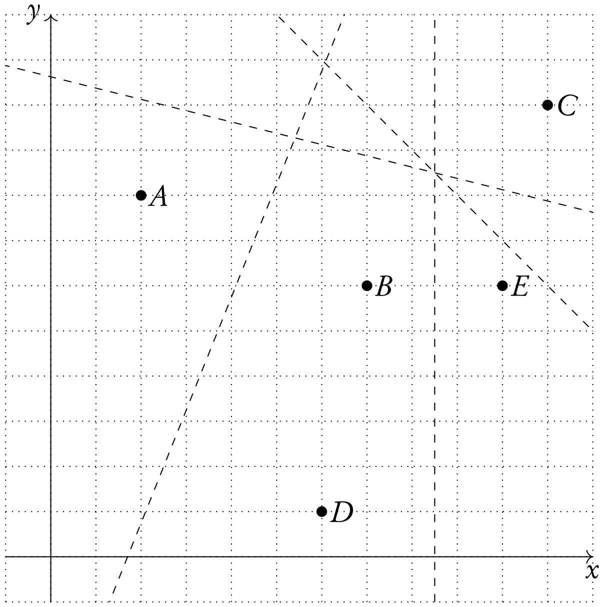
To construct a Voronoi diagram based on a set of sites, you need to find the boundaries between the cells that will enclose each of them. These edges lie along the perpendicular bisectors between neighbouring sites. Finding these perpendicular bisectors is referred to as a form of nearest neighbour interpolation.

Perpendicular bisector the line passing through the midpoint between two points and at a 90°angle to the line segment that connects them

Nearest neighbour interpolation finding the site closest to any given point

Voronoi diagrams

Points $A(2,8)$, $B(7,6)$, $C(11,10)$, $D(6,1)$ and $E(10,6)$ represent restaurants of a pizza chain in a city.



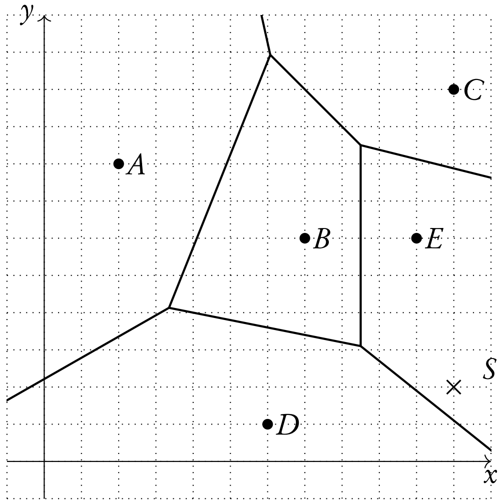
(a) Calculate the gradient of the line connecting points B and D.

(b) Hence, find the equation of the perpendicular bisector of points B and D.

(c) Sketch the completed Voronoi diagram.

To optimise delivery, pizza orders are always distributed to the closest restaurant.

(d) Sam lives at point $S(11,2)$. Which pizza place will prepare Sam's pizza?

1.	Use gradient formula to find m.	$m = \frac{6-1}{7-6} = 5$
2.	Find gradient of perpendicular line.	$m_2 = \frac{-1}{5}$ $\quad \Rightarrow y = \frac{-1}{5}x + c$
3.	Find one point that lies on the perpendicular bisector.	Midpoint of B and D: $\left(\frac{7+6}{2}, \frac{6+1}{2}\right) = (6.5, 3.5)$
4.	Use point to find c.	$3.5 = \frac{-1}{5}(6.5) + c$ $c = 4.8$ $\quad \Rightarrow y = \frac{-1}{5}x + 4.8$
5.	Sketch.	
6.	Use Voronoi diagram to determine which cell a given point falls into.	The pizza place closest to Sam is E, therefore Sam's order would be prepared at and delivered from E.

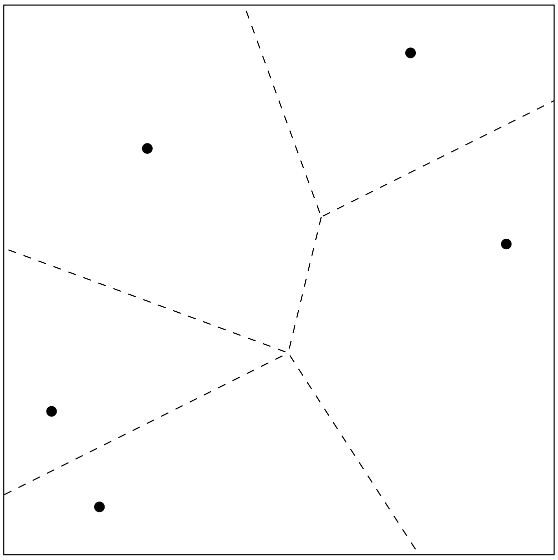
A perpendicular bisector of two points always passes through their midpoint

3.5.2 Applications

Voronoi diagrams can be used to model problems in for example ecology, meteorology or urban planning. You should be familiar with the so-called toxic waste dump problem.

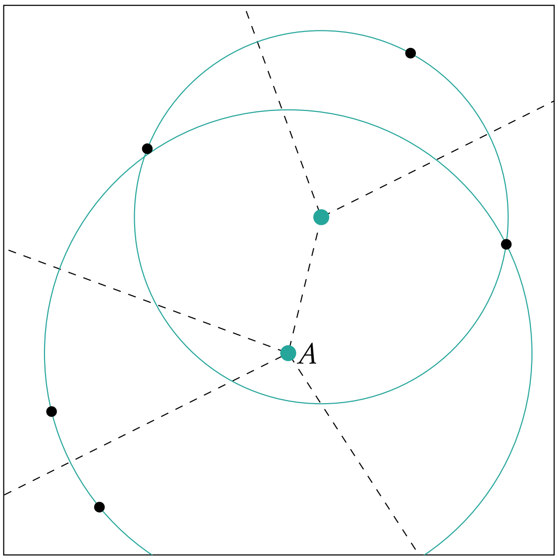
Example.

The following Voronoi diagram shows the centres of neighbouring urban hubs in a region.



What is the best place for the local government to place its toxic waste dump?

We assume that you want to place a toxic waste dump furthest away from any site. This is easy to find using a Voronoi diagram. You would place the dump on the vertex from which the radius to the nearest site is largest.



In this case, point A would be the best location for the region's toxic waste dump.

CALCULUS

4

Table of contents & cheatsheet

Definitions

Differentiation a way to find the gradient of a function at any point, written as $f'(x)$, y' and $\dfrac{dy}{dx}$.

Tangent line to a point on a curve is a linear line with the same gradient as that point on the curve.

Integration a way to find an original function from a derivative, written as $F(x)$, y and $\int f(x)\,dx$.

4.1. Differentiation 44

Polynomial a mathematical expression or function that contains several terms often raised to different powers

When $y = f(x) = ax^n$ then the derivative is $\dfrac{dy}{dx} = f'(x) = nax^{n-1}$.

Derivative of a constant (number) 0

Derivative of a sum sum of derivatives.

When $y = ax^n + bx^m$, $\dfrac{dy}{dx} = na^{n-1} + mbx^{m-1}$

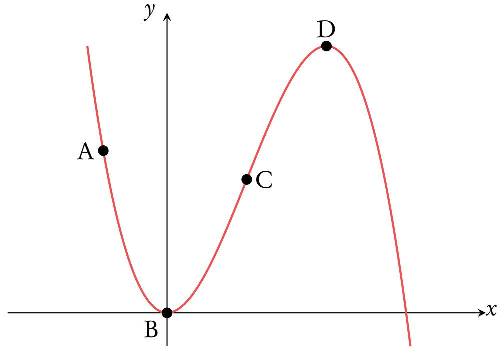
4.1.2. Tangent/Normal 46

Tangent a straight line that touches a curve at one single point. At that point, the gradient of the curve = the gradient of the tangent.

Normal a straight line that is perpendicular to the tangent line. Slope of normal $= \dfrac{-1}{\text{slope of tangent}}$

4.1.3. Turning points 49

Turning points occur when a function has a local maximum or local minimum. At these points $f'(x) = 0$.



	$f'(x)$	$f(x)$ is
A	$-$	decreasing
B	0	at local minimum
C	$+$	increasing
D	0	at local maximum

4.2. Integration 51

Indefinite integral

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

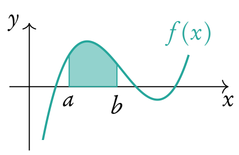
Definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where} \qquad F = \int f(x)\,dx$$

By determining a definite integral for a function, you can find the area beneath the curve that is between the two x-values indicated as its limits.

Area between curve and x-axis

$$A_{\text{curve}} = \int_a^b f(x)\,dx$$



Trapezoidal rule used to approximate the area under a curve

$$\int_a^b y\,dx \approx \frac{1}{2}h\left((y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right)$$

$$\text{where } h = \frac{b-a}{n}$$

4.1 Differentiation

As you have learnt in the unit on functions, a straight line graph has a gradient. This gradient describes the rate at which the graph is changing and using it we can tell how steep the line will be when plotted on a graph. In fact, gradients can be found for any function; the special thing about linear functions is that their gradient is always the same (given by m in $y = mx + c$).

Function	Gradient Function
$f(x)$	$f'(x)$
y	$\frac{dy}{dx}$

Non-linear functions however, will have changing gradients. Their steepness will be different at different x-values. This is where calculus comes in handy; we can use differentiation to derive a function using which we can find the gradient for any value of x. Two types of notation are used for calculus.

4.1.1 Polynomials

As functions forming curved lines, the gradients of polynomials are changing at each point. You can find the derivative function ($f'(x)$) for any polynomial function ($f(x)$) using the principles explained below.

Polynomial a function that contains one or more terms often raised to different powers

e.g. $y = 3x^2$, $y = 121x^5 + 7x^3 + x$ or $y = 4x^{\frac{2}{3}} + 2x^{\frac{1}{3}}$

Principles $y = f(x) = ax^n \Rightarrow \frac{dy}{dx} = f'(x) = nax^{n-1}$

DB 5.3

the (original) function is described by y or $f(x)$

the derivative (gradient) function is described by $\frac{dy}{dx}$ or $f'(x)$

Derivative of a constant (number) 0

e.g. for $f(x) = 5$, $f'(x) = 0$

Derivative of a sum sum of derivatives.

If a function you want to differentiate is made up of several summed parts, find the derivatives for each part separately and then add them together again.

e.g. $f(x) = ax^n$ and $g(x) = bx^m$

$$f'(x) + g'(x) = nax^{n-1} + mbx^{m-1}$$

When differentiating it is useful to first rewrite the polynomial function into a form that is easy to differentiate. Practically this means that you may need to use the laws of exponents before (or after) differentiation to simplify the function.

For example, $y = \frac{5}{x^3}$ seems difficult to differentiate, but using the laws of exponents we know that $y = \frac{5}{x^3} = 5x^{-3}$. Having the equation in this form allows you to apply the same principles as you would use to differentiate any other polynomial.

Example.

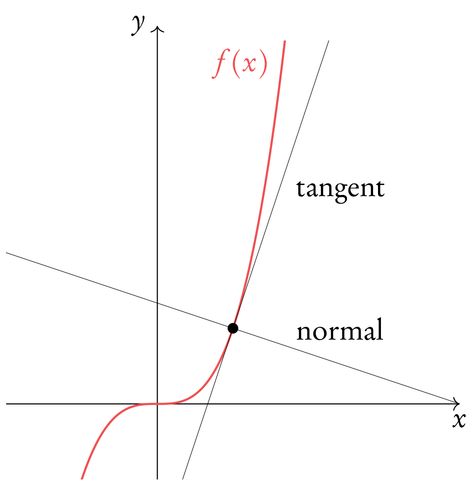
$f(x)$		$f'(x)$
5	$\longrightarrow$	0
x^2	$\longrightarrow$	$2\cdot 1x^{2-1} = 2x$
$4x^3$	$\longrightarrow$	$3\cdot 4x^{3-1} = 12x^2$
$3x^5 - 2x^2$	$\longrightarrow$	$5\cdot 3x^{5-1} - 2\cdot 2x^{2-1} = 15x^4 - 4x$
$\frac{2}{x^4} = 2x^{-4}$	$\longrightarrow$	$(-4)\cdot 2x^{-4-1} = -8x^{-5} = \frac{-8}{x^5}$
$3x^4 - \frac{2}{x^3} + 3$	$\longrightarrow$	$4\cdot 3x^{4-1} - 3\cdot(-2)x^{-3-1} + 0 = 12x^3 + \frac{6}{x^4}$

4.1.2 Tangent and normal equations

Tangent a straight line that touches a curve at one single point. At that point, the gradient of the curve is equal to the gradient of the tangent.

Normal a straight line that is perpendicular to the tangent line:

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}}$$



Finding the linear function of the tangent

Let $f(x) = x^3$. Find the equation of the tangent at $x = 2$

1.	Find the derivative and fill in value of x to determine slope of tangent.	$f'(x) = 3x^2$ $f'(2) = 3 \cdot 2^2 = 12$
2.	Determine the y value.	$f(x) = 2^3 = 8$
3.	Plug the slope m and the y value in $y = mx + c$.	$8 = 12x + c$
4.	Fill in the value for x to find c.	$8 = 12(2) + c$ $c = -16$ eq. of tangent: $y = 12x - 16$

Steps 1, 2 and 4 are identical for the equation of the tangent and normal

Finding the linear function of the normal

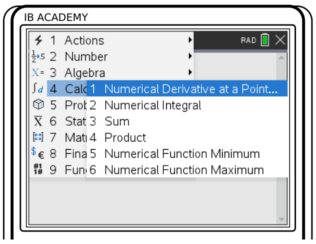
Let $f(x) = x^3$. Find the equation of the normal at $x = 2$

1.	____________________.	$f'(2) = 12$
2.	____________________.	$f(x) = 8$
3.	Determine the slope of the normal $m = \dfrac{-1}{\text{slope tangent}}$ and plug it and the y-value into $y = mx + c$.	$m = \dfrac{-1}{12}$ $8 = -\dfrac{1}{12}x + c$
4.	Fill in the value for x to find c.	$8 = -\dfrac{1}{12}(2) + c$ $c = \dfrac{49}{6}$ eq. of normal: $y = -\dfrac{1}{12}x + \dfrac{49}{6}$

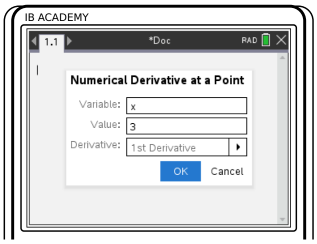
Steps 1, 2 and 4 are identical for the equation of the tangent and normal

To find the gradient of a function for any value of x.

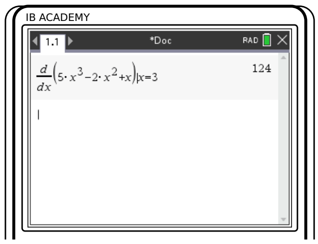
$f(x) = 5x^3 - 2x^2 + x$. Find the gradient of $f(x)$ at $x = 3$.



Press menu
`4: Calculus`
`1: Numerical Derivative at a Point`



Enter the variable used in your function (`x`) and the value of x that you want to find. Keep the settings on `1st Derivative`
Press OK



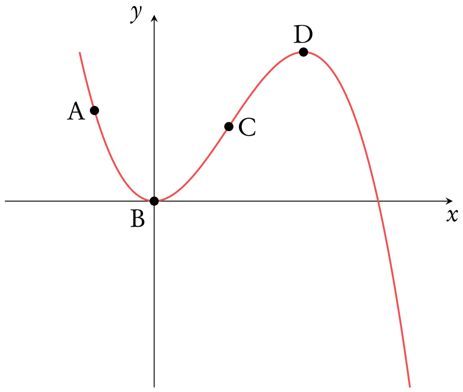
Type in your function
press enter

In this case, $f'(3) = 124$

4.1.3 Turning points

Turning points occur when a graph reaches a local maximum (top) or minimum (dip). This occurs when the derivative $f'(x) = 0$.

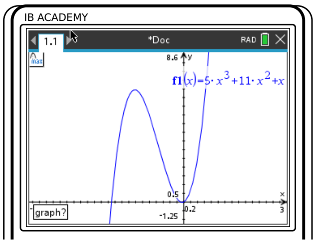
Use the graph (GDC) to see whether a turning point is a maximum or minimum.

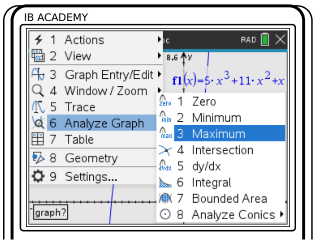


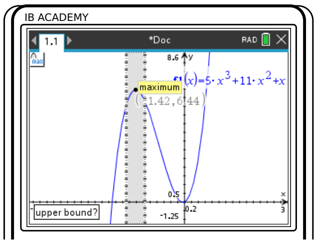
	$f'(x)$	$f(x)$ is
A	$-$	decreasing
B	0	at local minimum
C	$+$	increasing
D	0	at local maximum

Finding the coordinates of a turning point.

Let $f(x) = 5x^3 + 11x^2 + x$.
Find the coordinates of the local maximum point of $f(x)$.







Plot the function

Press (on), go to "Graph"
Enter the function
f1(x) $= 5x^3 + 11x^2 + x$

Find the local maximum point:

Press (menu)
`6: Analyze Graph`
`3: Maximum`

Approach the turning point you are trying to find with the cursor.
Click once to the left of the turning point and drag the selection to the right, beyond the turning point.
Click again.

Remember that you can find other attributes of a function using the `6: Analyse Graph` menu, for instance the gradient at any value of x with `5: dy/dx`

4.1.4 Optimisation

As we saw in the previous section, differentiation is useful for identifying maximum and minimum points of different functions. We can apply this knowledge to many real life problems in which we may seek to find maximum or minimum values; this is referred to as optimisation.

The most important thing to remember is that at a maximum or minimum point $f'(x) = 0$. So often if a question asks you to find a maximum/minimum value, just writing down $f'(x) = 0$ can score you points.

Determine the max/min value with certain constraints

The sum of the height h and base x of a triangle is 40 cm. Find an expression for the area in terms of x, hence find the maximum area of the triangle.

1. First write expression(s) for constraints followed by an expression for the actual calculation. Combine two expressions so that you are left with one variable.

$$x + h = 40$$
$$h = 40 - x$$
$$A = \frac{1}{2}xh = \frac{1}{2}x(40 - x) = -\frac{1}{2}x^2 + 20x$$

2. Differentiate the expression.

$$\frac{dA}{dx} = -x + 20$$

3. The derivative $= 0$, solve for x.

$$-x + 20 = 0$$
$$x = 20$$

4. Plug the x value into the original function.

$$A = -\frac{1}{2}(20)^2 + 20(20) = -200 + 400 = 200\,\text{cm}^2$$

4.2 Integration

4.2.1 Indefinite integrals and boundary conditions

Integration is essentially the opposite of differentiation; it can also be referred to as anti-differentiation. The following equation shows how to integrate a polynomial function:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

DB 5.5

As you can see, every time you integrate the power on your variable will increase by 1; this is opposite of what happens with differentiation, when you subtract 1. Whenever you integrate you also **add +C** to this function. This accounts for any constant that may have been lost while differentiating.

Whenever you differentiate any constants that were in the original function, $f(x)$, become 0 in the derivative function, $f'(x)$.

In order to determine the value of C, you need to fill in a point that lies on the curve to set up an equation with which you can solve for C. This is called a boundary condition.

Finding indefinite integrals

Let $f'(x) = 12x^2 - 2$
Given that $f(-1) = 1$, find $f(x)$.

1.	Separate summed parts (optional).	$\int 12x^2 - 2 \, dx = \int 12x^2 \, dx + \int -2 \, dx$
2.	Integrate.	$f(x) = \int 12x^2 \, dx + \int -2 \, dx = \frac{12}{3}x^3 - 2x + C$
3.	Fill in values of x and $f(x)$ to find C .	Since $f(-1) = 1$, $4(-1)^3 - 2(-1) + C = 1$, $C = 3$. So: $f(x) = 4x^3 - 2x + 3$

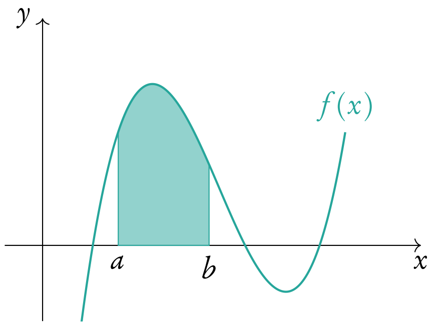
Note that this is the same thing you do when finding the y-intercept, c, for a linear function – see Functions: Linear models.

4.2.2 Definite integrals

If there are limit values indicated on your integral, you are looking to find a definite integral. This means that these values will be used to find a numeric answer rather than a function. The notation for definite integrals is:

$$\int_a^b f(x)\,\mathrm{d}x$$

A definite integral corresponds to the area of a region enclosed by the curve $y = f(x)$ and the x-axis, where $f(x) > 0$.



Finding definite integrals

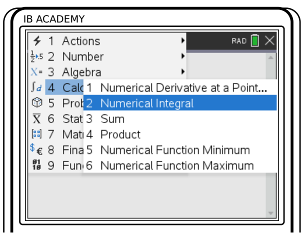
$$f(x) = \frac{1}{5}x^2 + \frac{1}{2}$$

Find the area enclosed between $f(x)$, the x-axis and the lines $x = 2$ and $x = 6$.

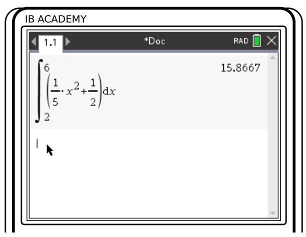
1. Write down the definite integral.

$$\int_2^6 \frac{1}{5}x^2 + \frac{1}{2}\,\mathrm{d}x$$

2. Solve using GDC.



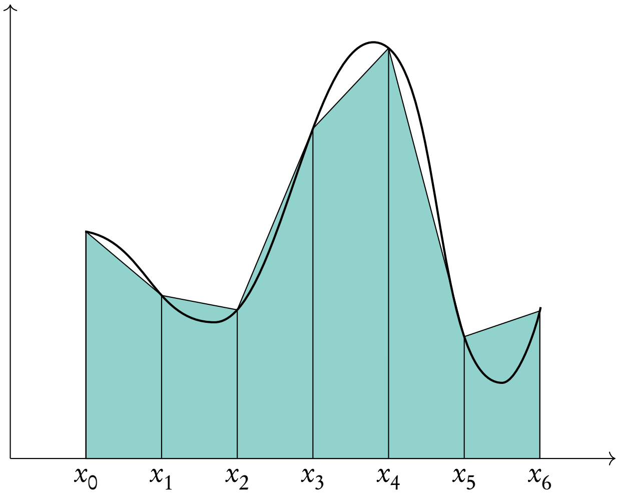
Press menu
4: Calculus
2: Numerical Integral



Fill out definite integral

So the area enclosed is 15.9 units^2

4.2.3 Trapezoidal rule



The trapezoidal rule is used to approximate the area under a curve. To make this approximation, we divide the area into several equal trapezoids.

The area of a single trapezoid can be found as $A = \frac{1}{2}(a+b)h$, where a and b are the parallel sides and h is the height.

The sum of the areas of the individual trapezoids serves as an approximation for the area under the curve. This can be found using the formula:

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\left((y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\right) \qquad \text{where } h = \frac{b-a}{n}$$

DB 5.8

Using trapezoidal rule

Approximate the definite integral $\int_2^{10} \frac{1}{2}x^2 + \frac{1}{2}\,\mathrm{d}x$ using the trapezoidal rule with 4 equal trapezoids.

1. Find h.

$$h = \frac{10-2}{4}$$
$$h = 2$$

2. Find all relevant y-values (these form the parallel sides of the trapezoids).

y_0 is the height at the lower boundary, y_n at the upper boundary.

In this case, we substitute the x-values at the trapezoid boundaries into $f(x)$. With 4 trapezoids, $x = 2, 4, 6, 8, 10$

y_n	$f(x)$	y-value
y_0	$f(2)$	2.5
y_1	$f(4)$	8.5
y_2	$f(6)$	18.5
y_3	$f(8)$	32.5
y_4	$f(10)$	50.5

3. Substitute values into formula.

$$\frac{1}{2}2\left((2.5+50.5)+2(8.5+18.5+32.5)\right)$$
$$= 172$$
$$\rightarrow \text{so } \int_2^{10} \frac{1}{2}x^2 + \frac{1}{2}\,\mathrm{d}x \approx 172$$

The exact value of this definite integral is 169.33 (to 2 d.p.), so this trapezoidal rule approximation gets us pretty close

PROBABILITY

5

Table of contents & cheatsheet

Definitions

Sample space the list of all possible outcomes.

Event the outcomes that meet the requirement.

Probability for event A, $P(A) = \dfrac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$.

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.

Conditional probability the probability of A, given that B has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

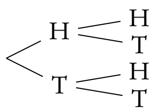
5.2. Multiple events 58

Probabilities for successive events can be expressed through tree diagrams or a table of outcomes.

Table of outcomes

	H	T
H	H,H	H,T
T	T,H	T,T

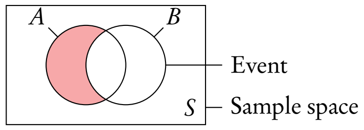
Tree diagram



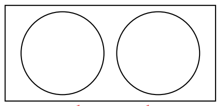
In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

5.1. Single events 56



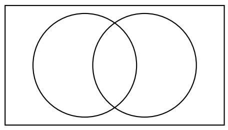
Mutually exclusive



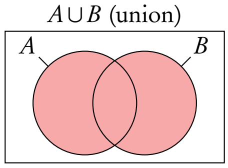
$P(A \cup B) = P(A) + P(B)$
$P(A \cap B) = 0$

Combined events

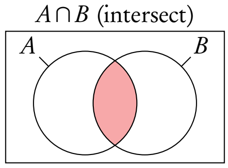
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$



$A \cup B$ (union)



$A \cap B$ (intersect)



If independent: $P(A \cap B) = P(A) \times P(B)$.

Compliment, A' where $P(A') = 1 - P(A)$

Exhaustive when everything in the sample space is contained in the events

5.3. Probability distributions 60

For a distribution by function the domain of X must be defined as $\sum P(X = x) = 1$.

Expected value $E(X) = \sum x P(X = x)$

Binomial distribution $X \sim B(n, p)$ used in situations with only 2 possible outcomes and lots of trials

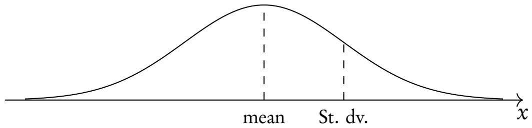
where n = number of trials, p = probability of success, r = number of success

On calculator:

- `binompdf`(n,p,r) $P(X = r)$
- `binomcdf`(n,p,r) $P(x \leq r)$

Mean $= np$

Normal distribution $X \sim N(\mu, \sigma^2)$



where μ = mean, σ = standard deviation

On calculator:

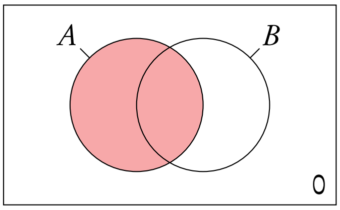
- `normcdf`(lower bound, upper bound, μ, σ)
- `invnorm`(area, μ, σ)

5.1 Single events

5.1.1 Venn diagrams

Probability for single events can be expressed through Venn diagrams.

DB 4.5



Sample space the list of all possible outcomes.

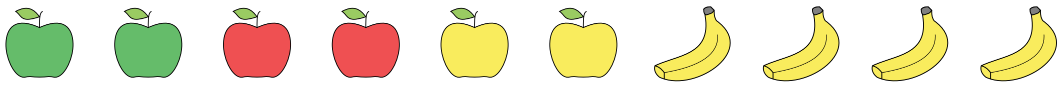
Event the outcomes that meet the requirement.

Probability for event A,

$$P(A) = \frac{\text{Number of ways } A \text{ can happen}}{\text{all outcomes in the sample space}}$$

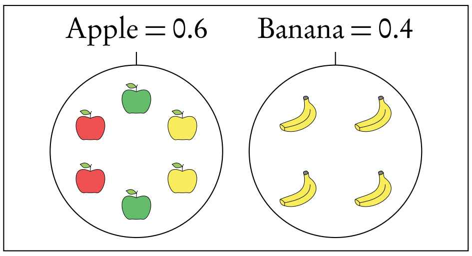
Here the shaded circle.

Imagine I have a fruit bowl containing 6 apples and 4 bananas.



Example.

I pick a piece of fruit.
What is the probability of picking each fruit?

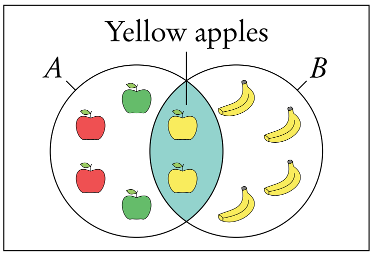


DB 4.6

As apples cannot be bananas this is mutually exclusive, therefore $P(A \cup B) = P(A) + P(B)$ and $P(A \cap B) = 0$. It is also an exhaustive event as there is no other options apart from apples and bananas. If I bought some oranges the same diagram would then be not exhaustive (oranges will lie in the sample space).

Example.

Of the apples 2 are red, 2 are green and 2 are yellow.
What is the probability of picking a yellow apple?



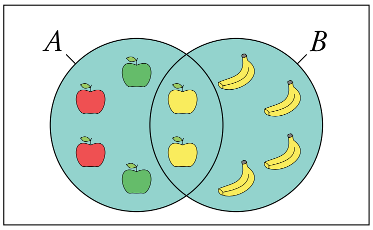
A: apples

B: yellow fruit

This is not mutually exclusive as both apples and bananas are yellow fruits. Here we are interested in the intersect $P(A \cap B)$ of apples and yellow fruit, as a yellow apple is in both sets $P(A \cap B) = P(A) + P(B) - P(A \cup B)$.

Example.

What is the probability of picking an apple or a yellow fruit?



A: apples

B: yellow fruit

This is a union of two sets: apple and yellow fruit.

When an event is exhaustive the probability of the union is 1.

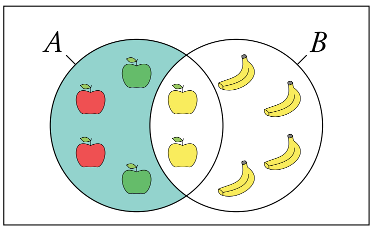
The union of events A and B is:

- when A happens;
- when B happens;
- when both A and B happen $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

DB 3.7

Example.

What is the probability of not picking a yellow fruit?



A: apples

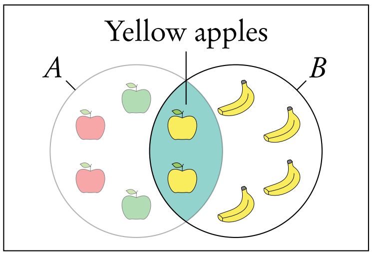
B: yellow fruit

This is known as the compliment of B or B'. $B' = 1 - B$.

Here we are interested in everything but the yellow fruit.

Example.

What is the probability of picking an apple given I pick a yellow fruit?

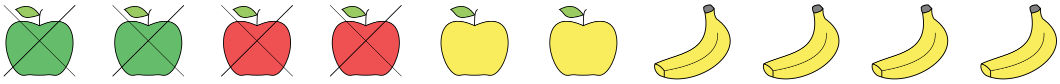


A: apples

B: yellow fruit

This is "conditional" probability in a single event. **Do not use the formula in the formula booklet.** Here we are effectively narrowing the sample space $= \dfrac{0.2}{(0.2+0.4)} = \dfrac{1}{3}$.

You can think of it like removing the non yellow apples from the fruit bowl before choosing.



Conditional probability $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

5.2 Multiple events

Independent events two events are independent if the fact that A occurs does not affect the probability of B occurring.
For independent events $P(A \cap B) = P(A) \times P(B)$

Dependent events two events are dependent if the outcome of event A affects the outcome of event B so that the probability is changed.

Conditional probability used for successive events that come one after another (as in tree diagrams). The probability of A, given that B has happened: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Questions involving dependent events will often involve elements that are drawn "without replacement". Remember that the probabilities will be changing with each new set of branches.

5.2.1 Tree diagrams

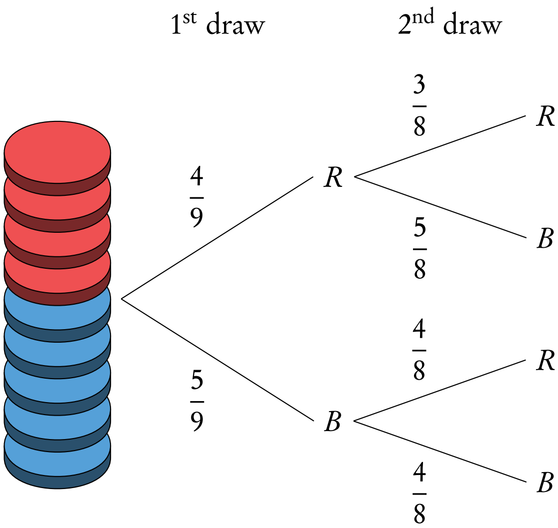
Probabilities for successive events can be expressed through tree diagrams. In general, if you are dealing with a question that asks for the probability of:

- one event **and** another, you **multiply**
- one event **or** another, you **add**

Example.

Two disks are randomly drawn without replacement from a stack of 4 red and 5 blue disks. Draw a tree diagram for all outcomes.

The probability of drawing two red disks can be found by multiplying both probabilities of getting red $\left(\frac{4}{9}\times\frac{3}{8}\right)$.



R and B: $\frac{4}{9}\times\frac{3}{8}=\frac{12}{72}$

R and B: $\frac{4}{9}\times\frac{5}{8}=\frac{20}{72}$

B and R: $\frac{5}{9}\times\frac{4}{8}=\frac{20}{72}$

B and R: $\frac{5}{9}\times\frac{4}{8}=\frac{20}{72}$

The probabilities for each event should always add up to 1. The probabilities describing all the possible outcomes should also equal 1 (that is, the probabilities that we found by multiplying along the individual branches).

What is the probability to draw one red and one blue disk?

P(one red **and** one blue)

$(P(R)$ and $P(B))$ **or** $(P(B)$ and $P(R))$

$(P(R)\times P(B))$ $\quad (P(B)\times P(R))$

$$\frac{20}{72}+\frac{20}{72}=\frac{40}{72}=\frac{5}{9}$$

It is common for conditional probability questions to relate to previous answers.

What is the probability to draw at least one red disk?

P(**at least** one red)

$P(R \text{ and } R)+P(B \text{ and } R)+P(R \text{ and } B)=1-P(B \text{ and } B)$

$$\frac{12}{72}+\frac{20}{72}+\frac{20}{72}=1-\frac{20}{72}=\frac{52}{72}=\frac{13}{18}$$

What is the probability of picking a blue disc given that at least one red disk is picked?

$$P(\text{blue disk} \mid \text{at least one red disk})=\frac{P(\text{a blue disk})}{P(\text{at least one red disk})}=\frac{\frac{5}{9}}{\frac{13}{18}}=\frac{10}{13}$$

5.3 Probability distributions

5.3.1 Discrete random variables

Another way of representing multiple events is with sample space diagrams. These show all the possible combinations of events in a sample space. The probabilities of the possible outcomes are often summarised in a table.

Once tabulated we can use the probability distribution to find the expected value. It is best to think of this as the average value you would get if you repeated the action many times.

Probability distribution links each outcome of a statistical experiment with its probability of occurrence; usually presented as a table or function

Discrete random variable a variable measured from a random phenomenon that has a finite number of possible outcomes (e.g. the outcome of flipping a coin)

DB 4.7

Expected value $E(X) = \sum xP(X = x)$

Probability distributions

A fair coin is tossed twice, X is the number of heads obtained. Find the expected number of heads obtained on two throws of the coin.

1. Draw a sample space diagram.

	H	T
H	H, H	H, T
T	T, H	T, T

2. Tabulate the probability distribution.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The sum of $P(X = x)$ always equals 1

3. Find the expected value of X: $E(X)$.

$$E(X) = \sum xP(X = x)$$
$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

So if you toss a coin twice, you expect to get heads once.

5.3.2 Distribution by function

A probability distribution can also be given by a function.

The domain of X must be specified, as the sum of the probabilities must equal 1.

Probability distribution by function

$P(X=x)=k\left(\frac{1}{3}\right)^{x-1}$ for $x=1,2,3$. Find constant k.

1.	Use the fact that $\sum P(X=x)=1$ to set up an equation.	$k\left(\frac{1}{3}\right)^{1-1}+k\left(\frac{1}{3}\right)^{2-1}+k\left(\frac{1}{3}\right)^{3-1}=1$
2.	Simplify and solve for k.	$k+\frac{1}{3}k+\frac{1}{9}k=\frac{13}{9}k=1$. So, $k=\frac{9}{13}$.

5.3.3 Binomial distribution

Binomial distribution type of probability distribution used to calculate the probability of obtaining a certain number of successes in a given number of trials

Binomial distribution is used in situations with only 2 possible outcomes (e.g. success or failure) and lots of trials.

Using GDC

In your exam you will be expected to find probabilities from binomial distributions using your GDC. There are two different functions that you can use for this. For both you will need to know the number of trials (n), the probability of success (p) and the expected number of successes (r).

Use `Binompdf` (n, p, r) for questions asking for the probability of an exact outcome, $P(X=r)$.
Use `Binomcdf` (n, p, r) for questions asking for the probability of a range of consecutive values, $P(X \leq r)$.

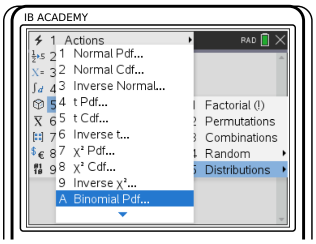
Note that by default `Binomcdf` only calculates $P(X \leq r)$ or in words "at most the value of r". Therefore you must remember to transform the function depending on the wording in the questions :

"Less than r" $P(X<r)=P(X \leq r-1)$
"More than r" $P(X>r)=1-P(X \leq r)$
"At least r" $P(X \geq r)=1-P(X \leq r-1)$

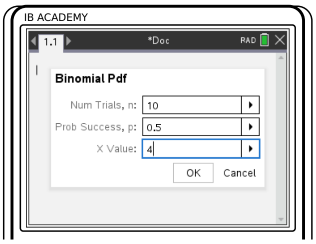
On some of the newer calculators you can specify what probability you are looking for, so this may not apply to you

Use binomial distribution to find a probability

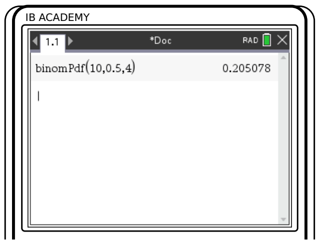
A fair coin is flipped 10 times. Find the probability of flipping 4 heads.



Press menu
`5: Probability`
`5: Distributions`
Select `Binompdf`



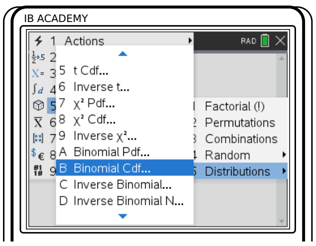
Enter values of
`n=10`
`p=0.5`
`r=4`
Press enter



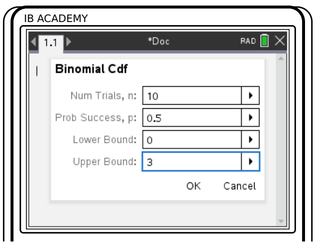
So the probability of getting exactly 4 heads is 0.205.

Use binomial distribution to find a probability

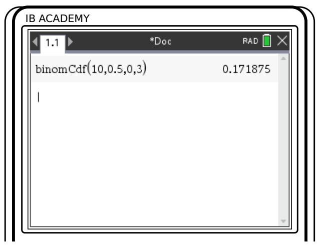
A fair coin is flipped 10 times. Find the probability of flipping less than 4 heads.
$P(X < 4) = P(X \leq 3)$



Press menu
`5: Probability`
`5: Distributions`
Select `Binomcdf`



Enter values of
`n=10`
`p=0.5`
`lower bound=0`
`upper bound=3`
Press enter

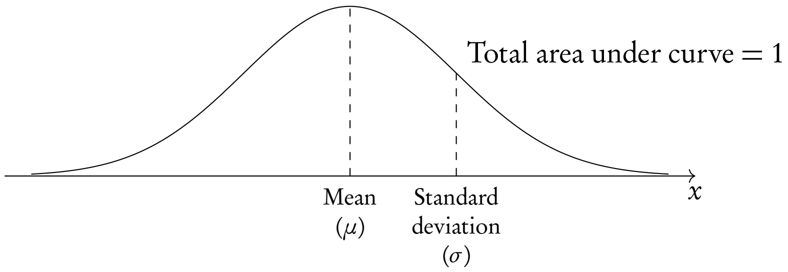


So the probability of getting less than 4 heads is 0.172.

5.3.4 Normal distribution

A normal distribution is one type of probability distribution which gives a bell-shape curve if all the values and their corresponding probabilities are plotted. The bell shape is symmetrical around the mean, μ. The width of the bell shape is given by the standard deviation, σ.

The IB expects you to be aware that 68% of the data lies between μ and $\pm\sigma$ (the mean and 1 st.dev. either side of it, 95% lies between μ and $\pm 2\sigma$, and 99% lies between μ and $\pm 3\sigma$.



We can use normal distributions to find the probability of obtaining a certain value or a range of values. This can be found using the area under the curve; the area under the bell-curve between two x-values always corresponds to the probability for getting an x-value in this range. The total area under the normal distribution is always 1. This is because the total probability of getting any x-value adds up to 1 (or, in other words, you are 100% certain that your x-value will lie somewhere on the x-axis below the bell-curve).

Using GDC

Use your GDC to answer questions dealing with normal distributions. You will either need to find probabilities for given x-values or x-values for given probabilities. In both cases, you will need to know the mean (μ) and standard deviation (σ) for the given example. These will be given in the question.

Use `normalcdf` (lowerbound, upperbound, μ, σ) for the probability that x is between any 2 values.

Even though you will be using your GDC, it's always useful to draw a quick sketch to indicate for yourself (and the examiner) what area or x-value you are looking for.

For lower bound $= -\infty$, use $-1\text{E}99$

For upper bound $= \infty$, use $1\text{E}99$

Use `invnorm` (ρ, μ, σ) to get an x-value for a given probability.

Most calculators assume that ρ is to the *left* of x. When the area/probability you are given is to the right of x, subtract it from 1 to get the ρ to use in `invnorm`.

Finding a probability/ percentage/ the area under a normal distribution curve

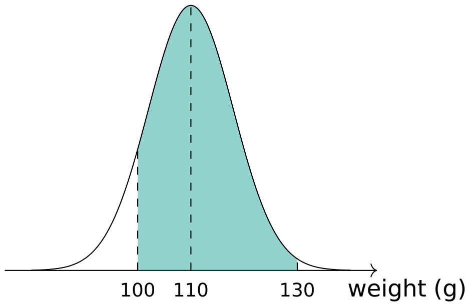
The weights of pears are normally distributed with mean $= 110$ g and standard deviation $= 8$ g.
Find the percentage of pears that weigh between 100 g and 130 g

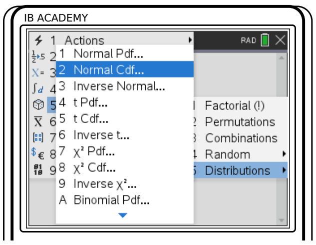
Sketch

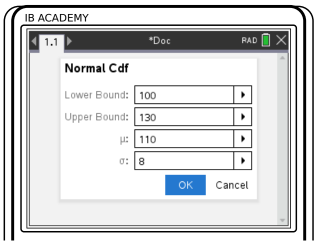
Indicate:

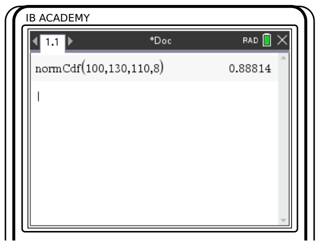
mean = 110 g
lower bound = 100 g
upper bound = 130 g

Shade area you are looking for









Press (menu), choose
`5: Probability`
`5: Distributions`
`2: Normal Cdf`

Enter lower and upper boundaries, mean (μ) and standard deviation (σ).
For lower bound $= -\infty$,
set `lower: -1E99`
For upper bound $= \infty$,
set `upper: 1E99`

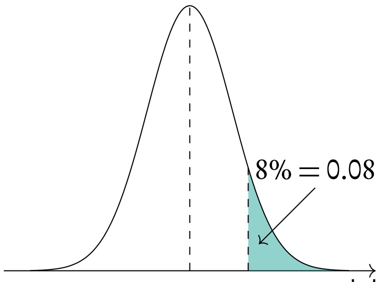
Press OK

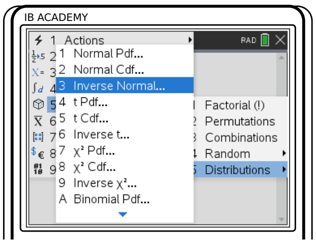
So 88.8% of the pears weigh between 100 g and 130 g.

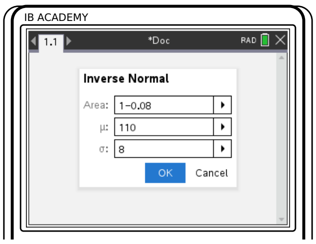
Finding an x-value when the probability is given

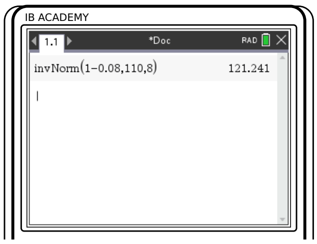
The weights of pears are normally distributed with mean $= 110\,\text{g}$ and standard deviation $= 8\,\text{g}$. 8% of the pears weigh more than m grams. Find m.

Sketch









Press menu
`5: Probability`
`5: Distributions`
`3: Inverse Normal`

Enter probability (`Area`), mean (μ) and standard deviation (σ).
The calculator assumes the area is to the left of the x-value you are looking for.
So in this case:
area $= 1 - 0.08 = 0.92$

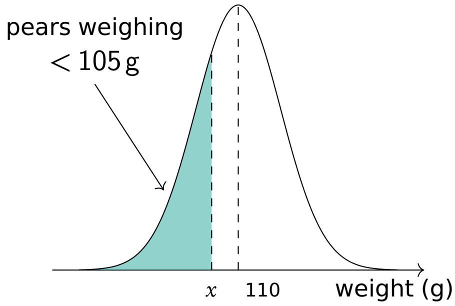
Press OK

So $m = 121$, which means that 8% of the pears weigh more than 121 g.

Finding an expected value

The weights of pears are normally distributed with $\mu = 110$ g and $\sigma = 8$ g. 250 pears are weighed. Find the expected number of pears that weigh less than 105 g.

1. Sketch.



2. Use `normcdf` to find the relevant probability.

lower bound = -1E99
upper bound = 105
μ = 110
σ = 8 $\rightarrow p$ = 0.266

3. Find expected value by multiplying the probability by the total number $E(x) = P(x) \times n$.

DB 4.8

expected number $= 0.266 \times 250$
$= 66.5$

$\rightarrow$ you would expect 67 pears to weigh less than 105 g

STATISTICS

6

Table of contents & cheatsheet

Definitions

Population the entire group from which statistical data is drawn (and which the statistics obtained represent).
Sample the observations actually selected from the population for a statistical test.
Random Sample a sample that is selected from the population with no bias or criteria; the observations are made at random.
Discrete finite or countable number of possible values (e.g. money, number of people)
Continuous infinite amount of increments (e.g. time, weight)

6.2. Descriptive statistics 69

For 1 variable data with frequency use `1-Var Stats` on GDC.

Mean $\bar{x} = \dfrac{\text{the sum of the data}}{\text{no. of data points}}$

Mode the value that occurs most often

Median the middle value when the data set is ordered low to high

Range largest x-value $-$ smallest x-value

Grouped data data presented as an interval
Use the midpoint as the x-value in all calculations.

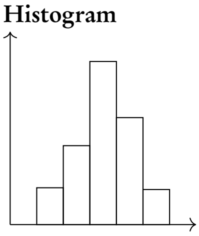
Q_1 first quartile $= 25^{\text{th}}$ percentile
Q_2 median $= 50^{\text{th}}$ percentile
Q_3 third quartile $= 75^{\text{th}}$ percentile
$Q_3 - Q_1$ interquartile range (IQR) $=$ middle 50 percent

6.2.2. Presenting and interpreting data 71

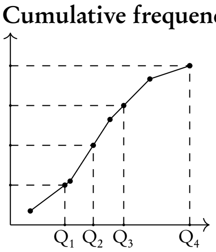
Frequency the number of times an event occurs in an experiment

Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

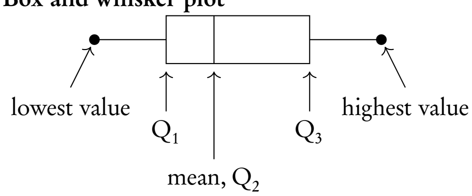
Histogram



Cumulative frequency



Box and whisker plot



6.3. Bivariate statistics 75

For analysis of data with two variables.

Pearson's correlation
Regression Line ($y = ax + b$) used to interpolate unknown data
Pearson's correlation coefficient (r) used to assess the strength of correlation

On GDC use `LinReg(ax+b)`.

Spearman's rank correlation
Spearman's rank correlation coefficient (r_s) used to assess the strength of a monotonic relationship between two variables

6.4. Chi-squared test 79

Chi-squared test of independence used to test independence of two variables

Chi-squared test of goodness of fit used to determine whether categorical data fit a hypothesized distribution

If $\chi^2 >$ critical value or p-value $<$ significance level (for 10% test, significance level $= 0.1$) *reject* null hypothesis (H_0).

6.5. T-test 83

For comparing data of two groups.

One-tailed t-test groups are compared in one direction from a reference direction (whether one mean is larger than the other **or** vice versa)

Two-tailed t-test groups are compared in both directions from a reference values (whether groups are significantly different, but not in which direction)

6.1 Basic statistical concepts

Note that continuous data can be presented as discrete data, e.g. if you round time to the nearest year or weight to the nearest kilogram.

Population the entire group from which statistical data is drawn (and which the statistics obtained represent).

Sample the observations actually selected from the population for a statistical test.

Discrete finite or countable number of possible values. (e.g. money, number of people)

Continuous infinite amount of increments. (e.g. time, weight)

6.1.1 Sampling techniques

In practice, samples can be obtained in different ways.

Simple random sample selected from the population with no bias or criteria; each member of the population has an equal chance of being picked

Convenience sample selected from a group that is easy to reach

Systematic sample selected at a fixed sampling interval; e.g. every 20^{th} customer is asked to fill out a questionnaire

Stratified sample a population is first split into several mutually exclusive subgroups. A random sample is then drawn from each subgroup.

Quota sample like a stratified sample, except a non-random sample is drawn from each subgroup (e.g. a convenience sample)

Sampling bias when not all members of a population have an equal chance of being picked; can result in a sample not being representative of the population (e.g. convenience samples are likely to be biased)

6.2 Descriptive statistics

The mean, mode and median, are all ways of measuring "averages". Depending on the distribution of the data, the values for the mean, mode, median and range can differ slightly or a lot. They are all useful for understanding your data set.

Example data set: 6, 3, 6, 13, 7, 7 in a table:

x	3	6	7	13
frequency	1	2	2	1

Mean the average value, $\bar{x} = \dfrac{\text{the sum of the data}}{\text{no. of data points}} = \dfrac{\sum x}{n} = \dfrac{\sum f x}{\sum f}$

DB 4.3

e.g. $\bar{x} = \dfrac{3+6+6+7+7+13}{6} = \dfrac{1\cdot 3+2\cdot 6+2\cdot 7+1\cdot 13}{1+2+2+1} = 7$

Mode the value that occurs most often (highest frequency)

e.g. The example data set has 2 modes: 6 and 7

Median the middle value when the data set is ordered low to high. Even number of values: the median is the average of the two middle values. Find for larger values as $n + \dfrac{1}{2}$.

e.g. data set from low to high: 3, 6, 6, 7, 7, 13

median $= \dfrac{6+7}{2} = 6.5$

Range largest x-value $-$ smallest x-value

e.g. range $= 13 - 3 = 10$

Standard deviation indicates the spread of a data set, σ *calculator only*

Grouped data data presented as an interval; e.g. $10 \le x < 20$ cm

- Use the midpoint as the x-value in all calculations. So for 10–20 cm use 15 cm.
- For 10–20 cm, 10 is the lower boundary, 20 is the upper boundary and the width is $20 - 10 = 10$.

Adding a constant to all the values in a data set or multiplying the entire data set by a constant influences the mean and standard deviation values in the following way:

	adding constant k	**multiplying by k**
mean	$\bar{x} + k$	$k \times \bar{x}$
standard deviation	σ	$k \times \sigma$

6.2.1 Quartiles

DB 4.2

Q_1 first quartile $= 25^{th}$ percentile.
The value of x so that 25% of all the data values are $\leq$ to it.

Q_2 median $= 50^{th}$ percentile

Q_3 third quartile $= 75^{th}$ percentile

IQR interquartile range $(Q_3 - Q_1)$ $=$ middle 50 percent

Outlier any point lower than $Q_1 - 1.5 \times \text{IQR}$ and larger than $Q_3 + 1.5 \times \text{IQR}$

Example.

Snow depth is measured in centimetres:
$30, 75, 125, 55, 60, 75, 65, 65, 45, 120, 70, 110$.
Find the range, the median, the lower quartile, the upper quartile and the interquartile range.

First always rearrange data into ascending order: $30, 45, 55, 60, 65, 65, 70, 75, 75, 110, 120, 125$

1. The range:
$$125 - 30 = 95\,\text{cm}$$
2. The median: there are 12 values so the median is between the 6^{th} and 7^{th} value.
$$\frac{65+70}{2} = 67.5\,\text{cm}$$
3. The lower quartile: there are 12 values so the lower quartile is between the 3^{rd} and 4^{th} value.
$$\frac{55+60}{2} = 57.5\,\text{cm}$$
4. The upper quartile: there are 12 values so the lower quartile is between the 9^{th} and 10^{th} value.
$$\frac{75+110}{2} = 92.5\,\text{cm}$$
5. The IQR
$$92.5 - 57.5 = 35\,\text{cm}$$

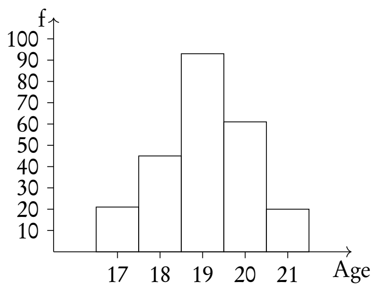
6.2.2 Presenting and interpreting data

Frequency the number of times an event occurs in an experiment

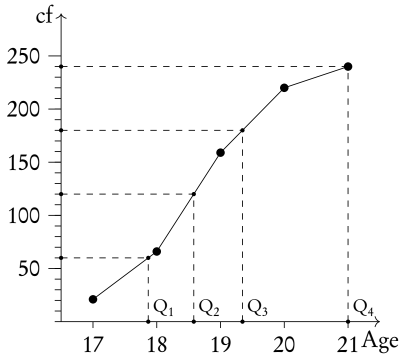
Cumulative frequency the sum of the frequency for a particular class and the frequencies for all the classes below it

Age	17	18	19	20	21
No. of students	21	45	93	61	20
Cumulative freq.	21	66	159	220	240

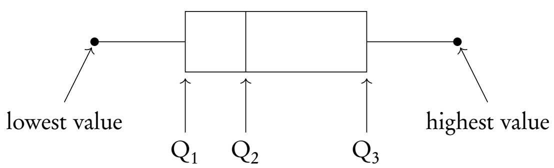
A cumulative frequency table summarises the cumulative frequencies for a data set.



A histogram is used to display and compare the frequencies for a specific condition. The frequencies (here: # of students) are displayed on the y-axis, and the different classes of the sample (here: age) are displayed on the x-axis. Neighbouring bars should be touching and their width should be drawn to scale (i.e. a wider class will be represented by a wider bar).



The cumulative frequency graph is used to display the development of the frequencies as the classes of the event increase. The graph is plotted by using the sum of all frequencies for a particular class, added to the frequencies for all the classes below it. The classes of the event (age) are displayed on the x-axis, and the frequency is displayed on the y-axis. The cumulative frequency graph always goes upwards, because the cumulative frequency increases as you include more classes.

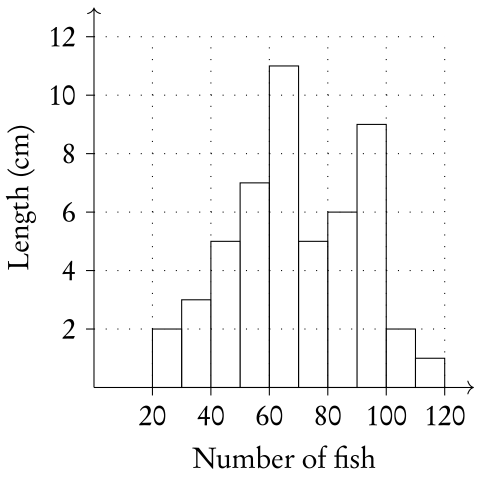


A box and whisker plot neatly summarises the distribution of a data set. It gives information about the range, the median and the quartiles. The first and third quartiles are at the ends of the box, the median is indicated with a vertical line inside the box, and the maximum and minimum points are at the ends of the whiskers. Box and whisker plots are drawn to scale in the x-direction

For your exam you will need to know how to find the value of Q_1, Q_2 and Q_3 using a cumulative frequency graph. First, determine the percentage of the quartile in question. Second, divide the total cumulative frequency of the graph (i.e. the total sample size) by 100 and multiply by the corresponding percentage. Then, you will have found the frequency (y-value) at which 25% for Q_1 / 50% for Q_2 / 75% for Q_3 of the sample is represented. To find the x-value, find the corresponding x-value for the previously identified y-value.

Example.

Using the histogram, create a cumulative frequency graph and use it to construct a box and whisker diagram.

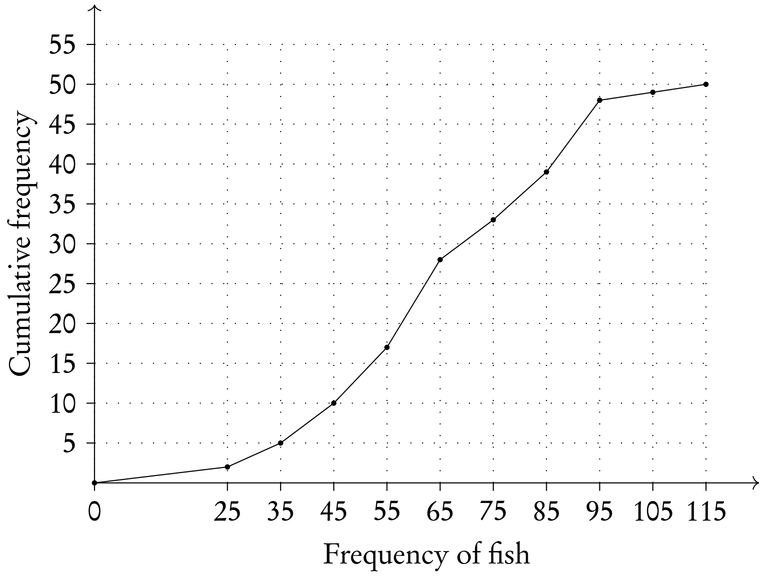


Write out the table for frequency and cumulative frequency.

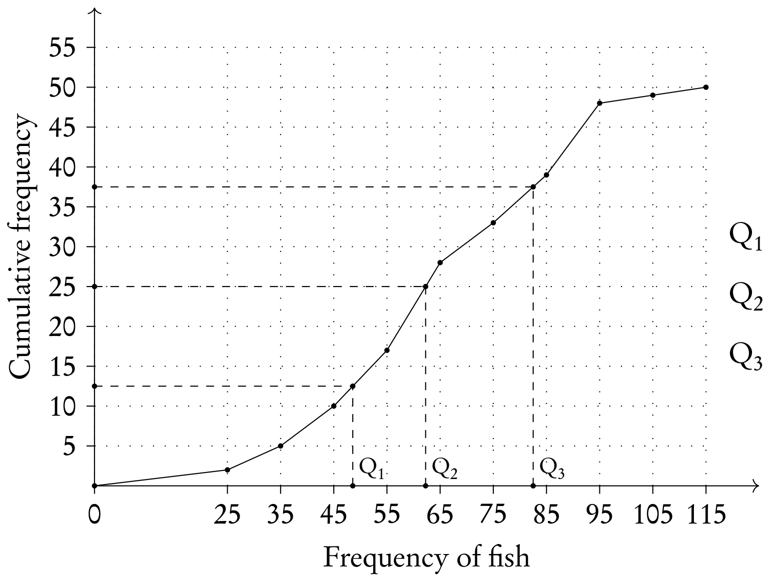
Frequency of fish	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Length of fish	2	3	5	7	11	5	6	9	1	1
Cumulative f.	2	5	10	17	28	33	39	48	49	50

Example.

Plot on cumulative frequency chart. Remember to use the midpoint of the date, e.g., 25 for 20–30.



Use graph to find Q_1, Q_2 and Q_3.

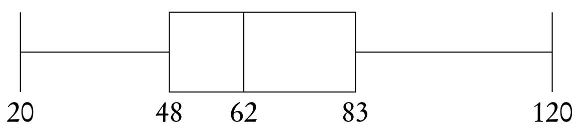


Q_1 25% of $50 = 12.5 \rightarrow 48$

Q_2 50% of $50 = 25 \rightarrow 62$

Q_3 75% of $50 = 37.5 \rightarrow 83$

Plot box and whiskers.

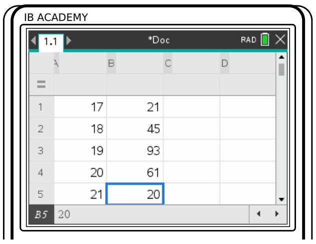


6.2.3 Using GDC

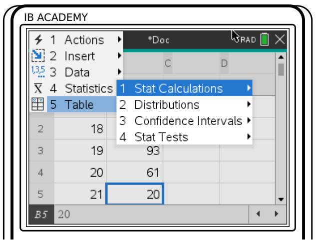
Finding the mean, standard deviation and quartiles etc.

Find the mean, standard deviation and Q1.

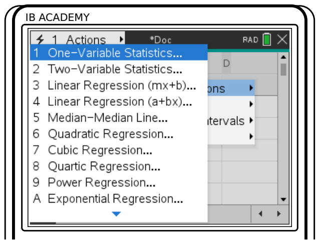
Age	17	18	19	20	21
No. of students	21	45	93	61	20



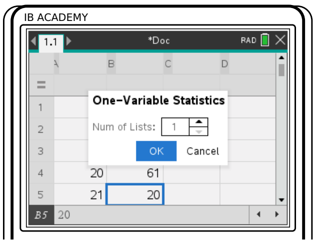
Press (on), go to `Lists and Spreadsheets`. Enter x-values in L1 and, if applicable, frequencies in L2



Press (menu), choose `4: Statistics` `1: Stat Calculations`

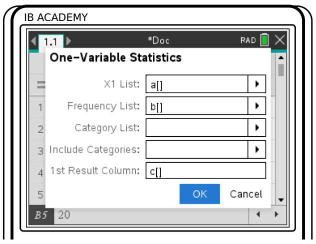


`1: One-Variable Statistics`



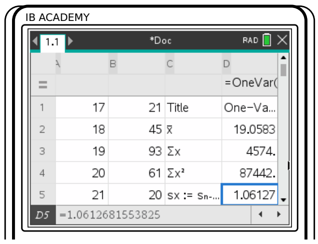
Enter `Num of lists: 1`.

Press OK



Enter names of columns you used to enter your x-list and frequency list and column where you would like the solutions to appear: `a[]`, `b[]` and `c[]`.

Press OK

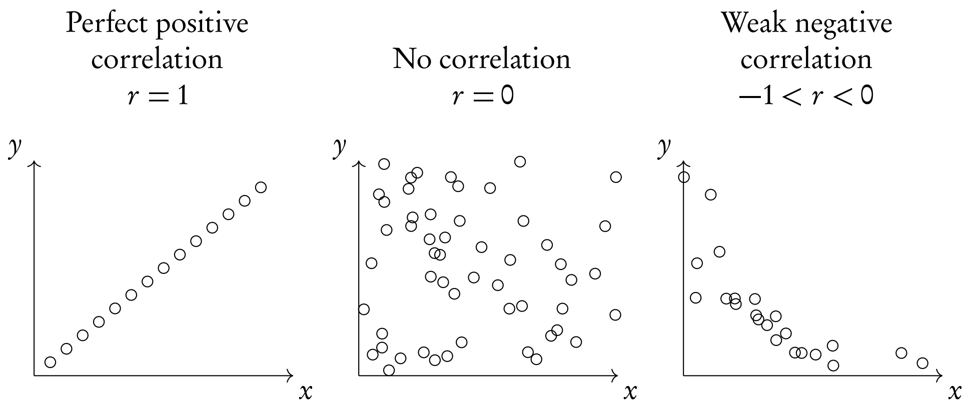


mean $= 19.1$
standard deviation $= 1.06$
Q1 $= 18.5$

6.3 Bivariate statistics

Bivariate statistics are about relationships between two different variables. You can plot your individual pairs of measurements as (x, y) coordinates on a scatter diagram. Analysing bivariate data allows you to assess the relationship between the two measured variables; we describe this relationship as correlation.

Scatter diagrams



Through statistical methods, we can predict a mathematical model that would best describe the relationship between the two measured variables; this is called regression. For your exam you will only have to focus on linear relationships, so only straight line graphs and equations. These so-called regression equations can be found using the GDC.

6.3.1 Pearson's correlation coefficient

Besides simply estimating the correlation between two variables from a scatter diagram, you can calculate values that will describe it in standardised ways. One of these is Pearson's correlation coefficient (r).

Pearson's correlation coefficient used to assess the strength of a linear relationship between two variables ($-1 \leq r \leq 1$)

$r = 0$ means no correlation.
$r = \pm 1$ means a perfect positive/negative correlation.

Interpretation of r-values:

r—value	$0 < \lvert r \rvert \leq 0.25$	$0.25 < \lvert r \rvert \leq 0.50$	$0.50 < \lvert r \rvert \leq 0.75$	$0.75 < \lvert r \rvert < 1$
correlation	very weak	weak	moderate	strong

Remember that correlation $\neq$ causation.

Calculate r while finding the regression equation on your GDC. Make sure that `STAT DIAGNOSTICS` is turned `ON` (can be found in the `MODE` settings), otherwise the r-value will not appear.

When asked to "comment on" an r-value make sure to include both, whether the correlation is:

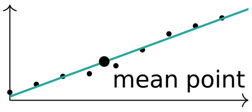
1. positive / negative
   and
2. strong / moderate / weak / very weak

Bivariate-statistics type questions

The height of a plant was measured the first 8 weeks

Week x	0	1	2	3	4	5	6	7	8
Height (cm) y	23.5	25	26.5	27	28.5	31.5	34.5	36	37.5

1. Plot a scatter diagram.



2. Use the mean point to draw a best fit line.

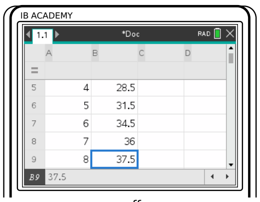
$$\bar{x} = \frac{0+1+2+\ldots+8}{9} = 3.56$$

$$\bar{y} = \frac{23.5+25+\ldots+37.5}{9} = 30$$

The line of best fit should pass through the mean point.

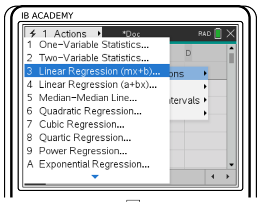
3. Find the equation of the regression line using GDC.

$y = 1.83x + 22.7$

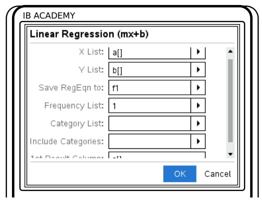


Press on (off), got to `“Lists and Spreadsheets”`

Enter x-values in one column (e.g A) and y-values in another column (e.g. B)

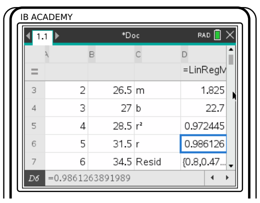


Press menu
`4: Statistics`
`1: Stat Calculations`
`3: Linear Regression (mx+b)`



Enter
`X list: A [];`
`Y list: B[];`
`1st Result Column: C[]`

Press OK



So, equation of regression line is $y = 1.83x + 22.7$ and Pearson’s correlation (r-value) $= 0.986$

4. Comment on the result.

Pearson’s correlation is $r = 0.986$, which is a strong positive correlation.

6.3.2 Spearman's rank correlation coefficient

Spearman's rank correlation coefficient used to assess the strength of a monotonic relationship between two variables ($-1 \leq r_s \leq 1$)

Monotonic relationship the values of two variables either both increase or both decrease, but not necessarily at a constant rate (i.e. can be linear or non-linear)

Spearman's rank correlation

Students' grades in Maths and Physics are recorded.

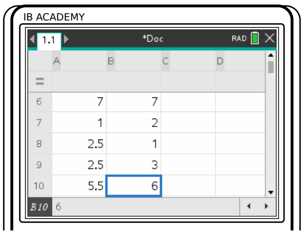
Maths	81	75	30	18	55	70	93	89	89	75
Physics	53	70	24	23	32	38	88	96	71	42

Find Spearman's Rank correlation coefficient and comment on it

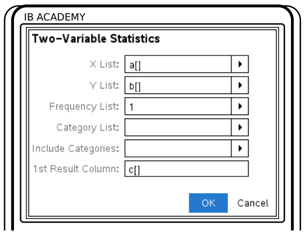
1. Fill out a table of ranks for all the data. The ranks of tied values are the means of the ranks they would have had if they were different.

Maths	4	5.5	9	10	8	7	1	2.5	2.5	5.5
Physics	5	4	10	8	9	7	2	1	3	6

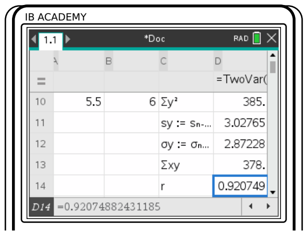
On TI-84, Spearman's Rank Coefficient can also be found using the RSX function



Press doc (+page)
4: Insert
6: Lists and Spreadsheets
Enter table of ranks into two columns (e.g A and B)



Press menu
4: Stats
1: Stat Calculations
2: Two-variable Statistics
Enter X and Y Lists



Press OK
$r_s = 0.920749$
So there is a strong positive relationship between students' Maths and Physics grades.

6.4 Chi-squared test

6.4.1 Independence

Chi-square test of independence used to test independence of two variables

H_0 the variables are independent (null hypothesis)

H_1 the variables are **not** independent (alternative hypothesis)

Determine if the variables are independent by the χ^2 test

	Directors	Managers	Teachers	Totals
Male	26	148	448	622
Female	6	51	1051	1108
Totals	32	199	1499	1730

Perform a χ^2 test of independence at the 10% significance level to determine whether employment grade is independent of gender.

1. State the null and alternative hypotheses.

H_0: gender and employment grade are independent
H_1: gender and employment grade are not independent

2. Calculate the table of expected frequencies

Formula: $\frac{tf_1}{T} \cdot \frac{tf_2}{T} \cdot T$.

e.g. expected number of male directors:

$$\frac{622}{1730} \cdot \frac{32}{1730} \cdot 1730 = 11.5$$

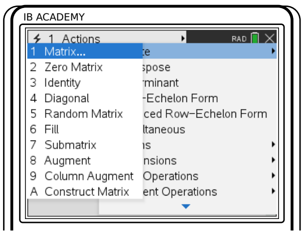
	Directors	Managers	Teachers
Male	11.5	71.5	539
Female	20.5	127.5	960

The expected frequency formula is not given in your data booklet, but you are expected to know it.

3. Write down the degrees of freedom df = (# rows - 1)(# columns - 1).

$\text{df} = (2-1) \cdot (3-1) = 2$

4. Write down the chi-square value using GDC..



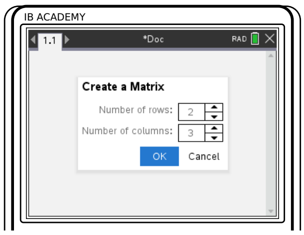
Enter data into GDC

Press menu

`7: Matrices & Vectors`

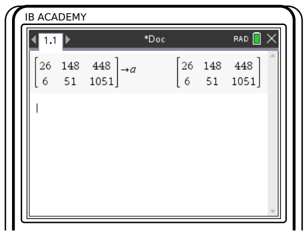
`1: Create`

`1: Matrix`



Enter dimensions of matrix to fit your data. Be sure you do *not* include the totals, so in this case you have a 2×3 matrix

Press OK

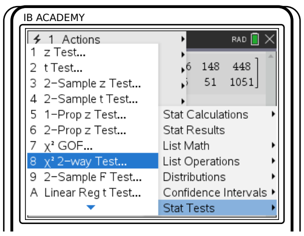


Enter the data as a matrix

press ctrl and var (sto→)

Give matrix a name by typing it after the arrow (e.g. a)

press enter

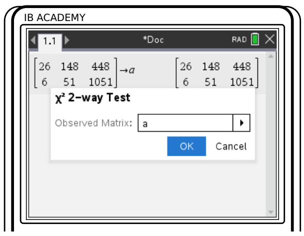


Press menu

`6: Statistics`

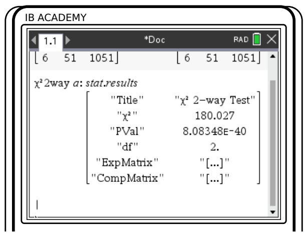
`7: Statistical Tests`

`8:` χ^2 `2-way Test`



Enter name of `Observed Matrix` (in this case a)

Press OK



So Chi-square value $\chi^2 = 180.03$, and p-value $= 8.08 \times 10^{-40}$

You will always be given the critical value in exam questions.

5. The critical value is 4.605. Use the critical value or the p-value to comment on your result.

Reject null hypothesis if:

χ^2 > critical value

or

p-value < significance level.

180.03 > 4.605

→ so we reject the null hypothesis; gender and employment grade are not independent.

6.4.2 Goodness of fit

Chi-square goodness of fit test used to determine whether categorical data fit a hypothesized distribution

H_0 the variables are consistent with the hypothesized distribution (null hypothesis)

H_1 at least one of the categories is **not** consistent with the hypothesized distribution (alternative hypothesis)

Determine goodness of fit of variables using the χ^2 test

Chocolate bar	Number sold
County	1526
Lars	758
Swix	1780
Silkyway	847
Trickers	1089
Total	6000

Perform a χ^2 test of goodness of fit at the 5% significance level to determine whether chocolate bar sales fit a uniform distribution.

1. State the null and alternative hypotheses.

H_0: chocolate bar sales are uniformly distributed
H_1: at least one of the chocolate bar sales does not fit a uniform distribution

2. Find the expected frequencies.

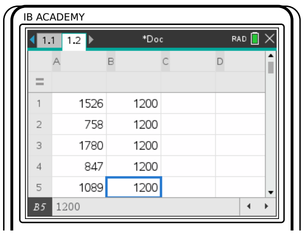
Uniform distribution means that each category is equally likely.
With 5 types of chocolate bar and 6000 sales recorded in total, the expected frequency for each category:

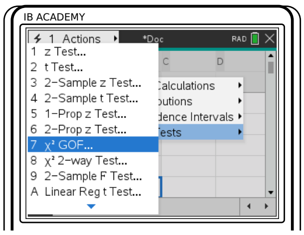
$$\frac{6000}{5} = 1200$$

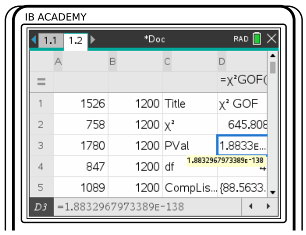
3. Write down the degrees of freedom $\text{df} = (\#\ \text{rows} - 1)$.

$\text{df} = (5 - 1) = 4$

4. Write down the chi-square value using GDC..







Enter data into GDC

Press on (off) go to `Lists and Spreadsheets`

Enter observed and expected data in two columns (e.g. `A` and `B`)

So Chi-square value $\chi^2 = 645.808$, and p-value $=$ 4.44089×10^{-16}

Press menu

`4: Statistics`

`4: Stat Tests`

`7:` χ^2 `GOF Test`

Enter name of `Observed` and `Expected` columns (in this case `A` and `B`)

Enter degrees of freedom

Press OK

5. Comment on your result.

$p < 0.05$

$\longrightarrow$ so we reject the null hypothesis; chocolate bar sales are not uniformly distributed.

6.5 T-test

The t-test is another type of statistical test that can be used to compare two groups. To apply a t-test, the variables you are testing should be normally distributed. Generally speaking, you can assume that this will be the case in exam questions that ask you to conduct a t-test.

***t*-test** tells you whether there is a significant difference between two groups by comparing their means

One-tailed test statistical significance is assessed only in one direction from a reference value (i.e. whether one mean is larger than the other *or* vice versa)

Two-tailed test statistical significance is assessed in both directions from a reference value (i.e. you test whether two groups are significantly different, but not in which direction)

Although there are several different t-tests (and different names for them!), we address only the type that you will need for your exam. This is also referred to as an Independent Samples t-test, Student t-test or unpaired t-test.

Depending on what a question or task is asking of you, you will have to determine whether to use a one- or two-tailed test.

Conduct a two-sample t test

A popcorn producer wants to compare the effect of two different fertilisers on the number of usable corn kernels yielded. They measure the average number of usable kernels per corncob in a random sample taken from two batches, each treated with a different fertiliser.

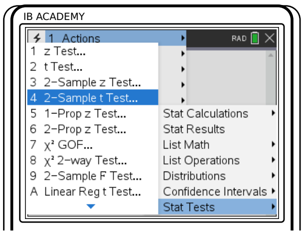
	Fertiliser A	Fertiliser B
Mean	768	803
Standard deviation	55	47
n	100	100

Use a t-test to determine whether the two fertilisers have a significantly different effect on corn kernel yield.

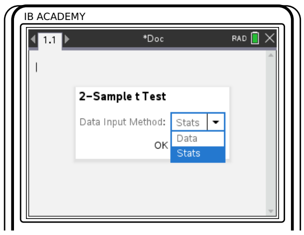
1. State the null and alternative hypotheses.

H_0: the means of kernels treated by fertiliser A and fertiliser B are equal ($\mu_1 = \mu_2$)
H_1: the means of kernels treated by fertiliser A and fertiliser B are not equal ($\mu_1 \neq \mu_2$)

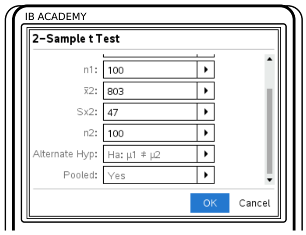
2. Conduct t-test using GDC.



Press menu
`6: Statistics`
`7: Stat Tests`
`4: 2-Sample t Test`

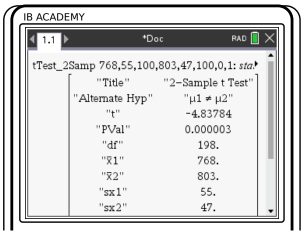


Select `Stats`.
OK



Enter means ($\bar{x}$), standard deviations (Sx) and sample sizes (n)
Select appropriate H_1 (H_a)
Choose '`Yes`' on Pooled
OK

As a general rule, you will always conduct a 'pooled' two-sample test. With a pooled test, we assume that the variance of the two samples is the same.



So $t = 4.838$, and
p-value < 0.00001

3. Comment on your result.

$p < 0.05$
$\longrightarrow$ so we reject the null hypothesis; the mean numbers of kernels yielded from batches treated by the fertilisers A and B are not equal.
Note that this two-tailed test does not tell us which fertiliser has a higher yield, just that there is a difference.

In your exam you will always be expected to use the p-value to compare the means of populations and draw conclusions from a t-test